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Robust Analytical and Computational Explorations of Coupled Economic-Climate Models with Carbon-Climate Response

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Abstract

The economics of global climate change is characterized by fundamental uncertainties, including the appropriate reduced forms for climate dynamics, the specification of economic damages resulting from climate change, and mechanisms by which these damages will affect long-run economic growth. Using a dynamic integrated assessment framework, this paper makes several contributions to improving the analysis of these uncertainties. First, we incorporate the cumulative climate response (CCR) function developed by Matthews et al. for representing the basic relationship between anthropogenic carbon emissions and increases in global mean temperature in a manner that is more directly policy relevant than the usual approach based on the equilibrium climate sensitivity. Second, we adapt the tools developed by Hansen, Sargent and others for robustness analysis to address underlying model uncertainty in both economic and climate dynamics. Third, we allow climate change to affect economic growth directly, in addition to its effect on output. We develop and study a simple analytical model that yields insights and results on the key implications of these assumptions, as well as facilitating the interpretation of numerical results from a more general model. Among our findings is that the presence of robustness may result in either a decrease or increase in the optimal carbon tax and energy usage, depending among other factors on societal preferences.

Keywords: Environment, Climate-Change, Model Uncertainty, Dynamic General Equilibrium.

JEL classification: Q32, Q43, and D81.

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1 Introduction

Uncertainty is the hallmark of global climate change and the analysis of policies to address it. The basic physical principles governing the response of the planetary atmosphere to increasing concentrations of greenhouse gases (GHGs) have been known since the nineteenth century. However, the capacity of numerical general circulation (climate) models to accurately predict the future course of the global climate system over multiple decades or longer is very limited, and subject to significant intra- and inter-model uncertainty. In the economics of climate change, there has been considerable debate regarding appropriate principles for analyzing the costs and benefits of GHG abatement, particularly regarding the problem of discounting over the very long run. Nonetheless, even conditional on discounting and other assumptions, cost-benefit analysis of abatement strategies using integrated assessment models - representing both economic and atmospheric dynamics - continues to be subject to extreme uncertainty, a point made forcefully by Pindyck (2013*b*)

Using a dynamic optimization framework, this paper makes two contributions to the economic analysis of climate change under uncertainty:

(i) To represent stylized climate dynamics at the global level of aggregation, we adopt the "Carbon Climate Response (CCR)" measure of Matthews et al. (2009), Matthews, Solomon and Pierrehumbert (2012), rather than the commonly-used climate sensitivity parameter. This enables us to focus the climate dynamics uncertainty that is relevant for policy making on the CCR parameter, not on the usual climate sensitivity parameter.

(ii) We adapt the approach of Hansen et al. (2006) to introduce model uncertainty and analyze robustness with respect to both climate and economic dynamics;

The definition and estimation of the CCR is motivated by the goal of narrowing the manifold uncertainties attending the application of climate science to the formulation of GHG abatement policies. To-date, a considerable share of GHG policy analysis has focused on the achievement of atmospheric carbon dioxide concentration targets. However, Matthews, Solomon and Pierrehumbert (2012) draw three conclusions from the climate science literature. First, the relationship between the flow of GHG emissions over time, and the resulting atmospheric concentration, is complex and difficult to model for policy purposes. Second, the relationship between concentrations and temperature changes - which the climate sensitivity summarizes - is poorly understood; as they point out, several decades of research has yielded a very small reduction of uncertainty about the magnitude of this quantity. Third, the effect of achieving a given concentration target on eventual equilibrium climate change occurs with a very long lag - multiple centuries - far beyond reasonable policy planning horizons.

By contrast, they infer from the results of climate model simulations and empirical evidence that the relationship between cumulative GHG emissions and global temperature change - the CCR - can be well-quantified, and its uncertainty measured with reasonable precision. The basis is the finding that temperature change $T_t - T_0$ from a starting date zero to a date t - e.g., from the pre-industrial era to the present - that is induced by injections of carbon into the atmosphere is roughly a linear function of cumulated injections, $\int_{s=0}^{t} bE_s ds$, where E_s is the flow of energy units used by the economy at date s and b converts units of energy into units of atmospheric carbon. Moreover, Matthews et al. (2009) report that the uncertainty in their estimates of the CCR is much less than current estimates of the uncertainty in the climate sensitivity, the distribution of which has been shown to have a fat tail Roe and Baker (2007), Roe (2013*b*), a property highlighted by Weitzman (2011), Pindyck (2011), and many others. Thus, the CCR is a well-grounded alternative (to the climate sensitivity) means of summarizing the fundamental relationship between human activity and the global climate for purposes of economic analysis.

Our inclusion of robustness analysis is motivated in part by the empirical properties of the CCR. As shown in Figures 3a in Matthews et al. (2009), each of the 11 climate models the output of which was used in the analysis produces an approximately constant long run value of the CCR, but this value varies across models. The climate modeling community does not estimate or report model probabilities or weights that would allow a single distribution to be assign to this set of values. Computing such probabilities is very difficult and it is likely that any such approximation to the probabilities would contain large errors. Thus, robustness analysis is appropriate to incorporate the inter-model uncertainty in the long-run value of the

CCR. Hansen and Sargent (2008) give a detailed defense for using their approach to robustness analysis. While it is not the only approach to robustness analysis, we use it here because it is simple and very convenient for the analytical and computational approach we develop in this paper.

Note that, Figure 3b in Matthews et al. (2009) shows both the high initial variability in the CCR for individual models as well as the lengths of time required for approximate convergence to the model-specific long-run values. We can interpret this inter-model spread as a rough measure of the uncertainty displayed by these transients. Hence, it is also appropriate to introduce robustness analysis at a shorter time scale to capture this type of uncertainty.

To provide context for our analysis, we next provide a brief overview of relevant previous work. Since their initial development more than two decades ago, integrated assessment (IA) models have been primarily deterministic. This is especially true of the high-dimensional versions, which link partial or general equilibrium models of the world economy with intermediate complexity climate models and other parts of the carbon cycle as well as ecosystem models. The economic components of these models are based on a calibration philosophy that does not in most instances include statistical procedures for parameterization and associated uncertainty quantification (Dawkins, Srinivason and Whalley (2001)). Moreover, the size of these large models generally precludes the use of stochastic optimization methods - a consequence of the "curse of dimensionality."

In parallel to the development of these large IA models, a substantial body of work has been conducted using lower-dimensional IA models following the Ramsey-Cass-Koopmans (RCK) optimal control framework. This research has to a very large extent been based directly or indirectly on the DICE (Dynamic Integrated Climate Economy) model of Nordhaus (2008), which has come to play a paradigmatic role in this field. Here too, most analysis has been deterministic. The tractability of the RCK approach, however, has facilitated various forms of stochastic analysis by a number of researchers. Following are a number of key examples.

Nordhaus and Popp (1997) developed the PRICE (PRobabilistic Integrated model of Climate and Economy) variation of DICE and used it to compare five methods of estimating the value of information regarding eight uncertain parameters, analyzed singly and jointly. Kolstad (1996) created and solved a stochastic version of DICE to analyze the influence optimal policy of learning about damages caused by climate change. Extending this work, Kelly and Kolstad (1999) implemented a stochastic variant of DICE, solved by dynamic programming, to conduct a Bayesian analysis of learning about the relationship between GHG levels and global mean temperature changes, in the presence of a stochastic shock to temperature. Keller, Bolker and Bradford (2004) adapted DICE to include a climate-related environmental threshold the collapse of the Atlantic thermohaline circulation due to temperature increase - learning, and uncertainty in the climate sensitivity, and solved this model using a global optimization method. Crost and Traeger (2011) developed a version of DICE in a recursive dynamic programming framework with uncertainty in damages and Epstein-Zin utility to study the different effects of risk, risk aversion, and aversion to intertemporal substitution. Jensen and Traeger (2013) use the stochastic DICE framework to study how uncertainty in long-run economic growth affects optimal climate policy.

The most ambitious extension of a DICE type framework to the stochastic case is the work of Cai, Judd and Lontzek (2012*b*). Their climate sector has three layer carbon cycle dynamics and a two layer atmosphere and ocean temperature dynamics. When these state variables are added to the state variables from the economic dynamics, there are a total of 8 state variables. The Cai, Judd and Lontzek (2012*b*) model is solved by a sophisticated (and quick) optimization algorithm that they have developed which is quite readily adaptable to other dynamic models. Continuous time methods are treated in Cai, Judd and Lontzek (2012*a*). The Cai, Judd and Lontzek (2012*c*) paper extends their work to include abrupt changes in climate dynamics, e.g. tipping points and the impact this possibility has upon the solution of the model. Tipping points can be viewed as a form of catastrophic climate change and are, indeed, catastrophic, if they are large enough. Cai, Judd and Lontzek (2012*c*) also extends their work to the case of recursive preferences.[1] As the reader will see below, our robustness analysis work is on a much simpler and less realistic minimalist model that exploits very recent work of Matthews et al. (2009). More will be said

[1] In other settings researchers have noted a close relationship between recursive preferences and a specific formulation of robustness (cf. Anderson, Hansen and Sargent 2003, Maenhout 2004, Hansen and Sargent 2008). While we do not explore that connection here, the Cai, Judd and Lontzek (2012*c*) formulation and analysis may well provide a valuable platform for the study of the implications of robustness on the part of economic agents inside the model.

about this later.

In recent years, several researchers have introduced robustness and ambiguity aversion into integrated assessment modeling. Following Hansen and Sargent (2001), Hennlock (2009) and Hennlock and Sterner (2012) incorporate robustness with respect to uncertainty regarding the product of climate sensitivity and equilibrium radiative forcing, in a model with both clean and dirty energy sectors, both of which have a form of endogenous technical change. Lemoine and Traeger (2011) adapt DICE to include an uncertain tipping point and learning about the threshold that triggers it, and aversion to ambiguity regarding the thresholds distribution. See the work of Cai, Judd and Lontzek (2012*b*,*c*) above for much more extensive and thorough work on stochastic extensions of the DICE type framework with tipping points as well as recursive preferences. Li, Narajabad and Temzelides (2012) adapt the model of Golosov et al. (forthcoming), assuming that climate change directly damages i.e., reduces the capital stock, with include model uncertainty embodied in a stochastic parameter governing the magnitude of this effect, and analyze robustness with respect to this uncertainty in a dynamic two-person zero-sum game, pitting the social planner against a malevolent agent (who controls the capital stock damage).

The present work contributes to this recent literature. We also adopt the Golosov et al. model, but simplify it by abstracting from the different types of energy that they include, instead defining a single energy input to production that is constrained by a non-renewable resource stock. In addition to incorporating robustness regarding the dynamics of global mean temperature, we use the Golosov et al. exponential damage function, and include robustness with respect to that functions parameter. Furthermore, we also assume that climate change damages the economy by affecting the economys productivity growth rate, and incorporate robustness to this effect as well. This mechanism of potential climate impact on the economy has recently been studied by Moyer et al. (2013); our analysis is the first to also make this the focus of robustness analysis. (Li, Narajabad and Temzelides (2012) assume no technical change in their analytical model, and constant rates of exogenous technical change in their computational model; the economic growth-rate uncertainty in Jensen and Traeger (2013) is not subject to direct damage from climate change.)

As is true of most integrated assessment research using a dynamic optimization framework, our approach is stylized and abstracts from many important economic and climate details. Our view, however, is that this level of abstraction facilitates thorough understanding of the behavior of the model and its implications that becomes difficult or impossible for larger computational models. This approach may go part way to answering the calls of M. Granger Morgan [see pages 23 and 24 of the Royal Society Science Policy Centre report DES2915 (2013)] and Pindyck (2013*a*) for simpler approaches in IA modeling that emphasize the promotion of insight over the generation of large quantities of numerical output. In addition, our approach to robustness helps to identify where potential vulnerability points of worst cases may lie for policy makers on the long time horizons that are relevant for climate risk management.

In a broad-ranging critique of integrated assessment modeling, Stern (2013*b*) calls for a "new generation" of such models that would, among other improvements, be developed explicitly within a risk-management framework. The work we have presented here contributes to achieving that goal. Moreover, both Lord Stern's and Pindyck (2013*a*) incisive observations on the shortcomings of current IA models can be interpreted, from the perspective of robustness analysis, as identifying fundamental and currently irreducible model uncertainty: IA modeling at its present stage of development cannot rule out the presence of underlying model misspecification. The philosophy and analytical methods of robustness are therefore extremely well-suited to developing the new generation of models that Lord Stern envisions.

The paper is organized as follows. Section 2 introduces and analyzes a simple closed-form example, which is used to help us interpret the computational results of a similar model, but with less restrictive assumptions, treated in Section 3. In Section 2 we conduct a robustness analysis of the impact of uncertainty in the Carbon Climate Response parameter that plays a key role in the Matthews et al. (2009), Matthews, Solomon and Pierrehumbert (2012) approach that we use in this paper. Section 2 also introduces a channel by which an increase in global average temperature up to date t increases the "size" of the set of perturbations around the baseline model at future dates $t+1, t+2, \ldots$. Section 3 contains the statement of the less restrictive model and a collection of computational results. Section 4 closes the paper with a set of brief comments that summarize our results, provide appropriate qualifications, and identify opportunities and priorities for future research

2 A Simple Model

In this section we adapt the robust control approach of Hansen et al. (2006) and the Carbon-Climate Response (CCR) approach of Matthews et al. (2009), Matthews, Solomon and Pierrehumbert (2012) to build a minimalist economic-climate dynamic optimization model. As noted in the Introduction, this model will be used to identify "vulnerability points" in attempting to construct a climate management policy that works uniformly well over the wide range of uncertainties that have been identified on both the climate dynamics side and the economic dynamics side, and that we incorporate in a simplified form [2]. We make strong assumptions on this very simple minimalist model in order to obtain some analytical results to help exposit the detailed numerical work presented in Section 4 for a computational model with less restrictive assumptions. Consider the following discrete time robust control model,

$$J \equiv \max_{C,K,E} \min_{G} \mathrm{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\gamma}}{1-\gamma} + \frac{1}{2} \left[\sum_{i \epsilon \{a, aT, c, D\}} \frac{\beta}{\theta_i} G_{i,t}^2 \right] \right) + \mu_{R_0} (R_0 - \sum_{t=0}^{\infty} E_0) \right\} \tag{2.1}$$

s.t.

$$C_t + K_{t+1} = Y_t \equiv A_t \Omega_t K_t^{\alpha} E_t^{\nu} + (1-\delta) K_t \tag{1}$$

$$T_t = T_0 + \lambda b (R_0 - R_t) + \hat{T}_t \tag{2}$$

$$\hat{T}_{t+1} = (1-\kappa_T)\hat{T}_t + \sqrt{\epsilon}\sigma_T(-G_{ct} + e_{T,t+1}) \tag{3}$$

$$\log A_{t+1} - \log A_t = \mu_{a0} - \mu_{aT}(T_t - T_0) + \sqrt{\epsilon}\sigma_{aT}(T_t - T_0)(-G_{aTt} + e_{aT,t+1}) + \sqrt{\epsilon}\sigma_a(-G_{at} + e_{a,t+1}) \tag{4}$$

$$R_{t+1} - R_t = -E_t \tag{5}$$

where we assume the simple damage function

$$\Omega_t = \exp\left[-D_t(T_t - T_0) + \sqrt{\epsilon}\sigma_D(-G_{D,t-1} + e_{D,t})\right] \tag{2.2}$$

[2] Our analysis addresses, for example, the problem posed by Pindyck (2013*c*): "Pricing carbon when we don't know the right price."

and the minimizing agent may act to impact the damages dynamics as well as the temperature dynamics. Below we discuss various processes for the damage coefficient D_t. In 2.1 $A_t, C_t, K_t, Y_t, R_t, T_t$ and δ denote level of technology, aggregate consumption, aggregate capital stock, aggregate output, total reserves of fossil fuels, global planetary average yearly temperature, and depreciation of capital stock, respectively. Initial values given by history are $A_0, C_0, K_0, Y_0, R_0, T_0, \hat{T}_0$. We assume, $0 < 1 - \kappa_T < 1$. We have added the multiplier μ_{R_0} times the resource constraint to the objective function in 2.1. Of course the utility function, $\frac{C_t^{1-\gamma}}{1-\gamma}$ and production function $F(K, E) = K^\alpha E^\nu$, could be replaced by more general functions. For example a natural extension would be to specify utility as a function of consumption and climate quality, $U(C_t, Q(T - T_0))$ where climatic quality, $Q(T - T_0)$ falls as $T - T_0$ increases. A particular specification that is very tractable is $U(C_t, Q(T - T_0)) = \log(C_t) - L(T_t - T_0)$ where $L(T_t - T_0)$ is a convex increasing "cost" function.

In this problem the θ_i's for $i = \{a, aT, c, D\}$ are robustness parameters. As $\theta_i \downarrow 0$ for all i, the problem converges to a usual dynamic stochastic problem without robustness. As θ_i increases more worries are placed on the misspecification of the corresponding equation. For example, as θ_c increases the agent becomes increasingly worried about errors in his specification of the law of motion for $\hat{T}$ in equation 3.[3]

Global average temperature, $T_t = T_0 + \lambda b(R_0 - R_t) + \hat{T}_t$ is composed of two parts, a trend term which is impacted by the world's emissions of Green House Gases (GHG's), $T_0 + \lambda b(R_0 - R_t)$ and fluctuation about trend, $\hat{T}_t = (1 - \kappa_T)\hat{T}_{t-1} + \sqrt{\epsilon}\sigma_T(-G_{c,t-1} + e_{T,t})$, where the term $-\sqrt{\epsilon}\sigma_T G_{c,t-1}$ is absent in the non-robust case. The minimizing agent chooses the non-anticipating process $\{G_{ct}\}$ to minimize the objective function in 2.1.

We will use the following solution concept for the intertemporal zero sum dynamic game 2.1, 2.2, which we shall call "Commitment Intertemporal Nash Equilibrium": At date 1 the maximizing player chooses $\{C_t, K_{t+1}, E_t, t = 1, 2, \dots\}$ and the minimizing player chooses $\{G_{it}, t = 1, 2, \dots\}$ for $i \epsilon \{a, c, aT, D\}$ over the set of non-anticipating stochastic processes and commits to these choices forever, i.e. for $t = 1, 2, \dots$.

[3]In many papers by Hansen and Sargent, they focus on one robustness parameter which is the same for all i. In addition, they usually call the "robustness parameter" the inverse of what we call the robustness parameter in this paper. We choose a different terminology because in later sections we compute expansions around a problem in which all θ_i's are zero.

Each player takes the other player's intertemporal choice of non-anticipating stochastic process as given and beyond that player's control. We will solve below the First Order Necessary Conditions (FONC's) for the choices made by both the maximizing agent and the minimizing agent. We have not investigated uniqueness issues, and whether the solutions of the FONC's are local maxima and local minima (are global maxima and global minima). Our concept of equilibrium can be questioned because we have ignored possible time inconsistency problems by using this equilibrium concept. One can also question whether this concept captures the original motivation for formulating robustness as a solution of such a game, i.e., using the game as a mechanism for constructing a control that works well over a set of departures from a central "baseline" model.

Other approaches to formulating solution concepts include letting the minimizing agent choose $\{G_{it}, t = 0, 1, 2, \dots\}$ at date zero and committing to it forever and the maximizing agent then choosing $\{C_t, K_{t+1}, E_t, t = 0, 1, 2, \dots\}$. We shall call this formulation the "maximizing agent as Stackelberg leader case." Another possibility is to use Markov Perfect equilibria.

Hansen and Sargent (2008) discuss different concepts of equilibrium in these types of dynamic zero sum game approaches to robustness and prove invariance results to timing protocols in their linear quadratic settings. However, because we are not in the linear quadratic setting here, we must proceed with caution. Hansen et al. (2006) discuss these issues in general continuous time settings.

Since we are just using the closed form solutions we derive below under rather severe assumptions to help exposit the core computational results of this paper, therefore we content ourselves with simple FONC analysis of Nash commitment intertemporal equilibrium here.

The ϵ's appear in the formulas here and below in order to set the stage for analysis of small noise expansions that adapts the work of Anderson, Hansen and Sargent (2012). For example in some of the analysis to follow we will scale ϵ's and θ's so that $\epsilon \to 0, \theta \to \infty, \epsilon\theta \to \theta_0 > 0$ and reduce a stochastic robust control problem to a simpler "deterministic robust control" problem. We will also use some of the small noise methods in the computational Section 3 below.

The case $\gamma = 1$ for the utility function corresponds to, $U(C) = \log C$. We will work with this case for most of the analytical work in this paper. Other cases will be analyzed in the computational section 3 below. We will always assume $\gamma \geq 0$ in order to have a concave, increasing utility function.

The shocks $\{\epsilon\}$ can be quite general, e.g., mean reverting, or independently and identically distributed across time. We assume they are independent of each other with mean zero and finite variance, and when convenient we will also take the liberty of assuming they are normally distributed. We will sometimes put $D_{t+1} = d > 0, t = 0, 1, 2, \ldots,$ for simplicity.

Note that there are two channels for robustness to impact the dynamics of the damage function in 2.2 and there are two channels for robustness to impact the dynamics of technical change in 2.1. Thus, robustness impacts not only the level of aggregate output but also the rate of growth of output. The importance of uncertainty impacts on the rate of growth of output - not just its level - has recently been demonstrated by Moyer et al. (2013). Thus we may expect the combined impact of robustness on the rate of growth and level of output to be larger, potentially much larger, than the impact of robustness on the level of output alone.

The term b is the amount of carbon emitted into the atmosphere per unit of energy E_t used in the production process at date $t+1$. Units of E can be chosen so that $b = 1$. Note that Jensen and Traeger (2013) use Epstein-Zin recursive preferences to capture a prudence effect on optimal carbon taxes on growth uncertainty in a DICE-type setting. We are, here, focused on a direct channel where climate degradation can hurt the growth rate of the economy, as do Moyer et al. (2013). However, we just insert a "reduced form" representation of this channel into the growth equation for technical progress rather than modeling it explicitly as do Moyer et al. (2013). We take this crude approach to a growth rate damage channel in order to keep access to closed form solutions in this section of our paper. While we believe that climate degradation probably impacts the variance of growth, $\sqrt{\epsilon}\sigma_{aT}(T_t - T_0)$, we will assume that this variance is constant, i.e. $\sqrt{\epsilon}\sigma_{aT}(T_t - T_0) = \sqrt{\epsilon}\sigma_{aT}$ in what follows for simplicity. We emphasize that the model in this section is intended only for use in interpreting the computational results in Section 3.

Remark 2.1. *In this Remark, we develop robust "microfoundations" for variable θ's in 2.1. We wish to* *find controls that work uniformly well over the set*

$$\sum_{t=0}^{\infty} \beta_2^t \left[\sum_{i \epsilon \{a, aT, D, c\}} (G_{it} S_i)^2 \right] \leq \eta_0^2. \tag{R2.1}$$

We form the Lagrangian for problem 2.1 without the theta terms in the objective but attach the Lagrange multiplier for R2.1 times the constraint 2.1, i.e., we extremize this Lagrangian subject to the constraints in 2.1. Note that as η_0 increases that θ gets bigger (i.e. the penalty on the minimizing agent's choice gets smaller). The S_i's represent relative trust in the specification of the dynamics of sector $i, i \in \{a, aT, D, c\}$. If S_i is larger, we know more about, or trust more, the specification of the dynamics of sector i. Put $\theta_i = \theta / S_i$, for $i \in \{a, aT, D, c\}$. Thus, if we have no doubts at all about the specification of sector i dynamics, so that $S_i = \infty$, then $G_{it} = 0$ for all t and we use the convention that the term, $(G_{it} S_i)^2 = 0$ so that it is dropped from the summation in R2.1.

The use of "S's" here can be viewed as a simple way to treat structured uncertainty, i.e. to treat differences in the size of misspecification at different shock channels. That is, the "S's" approach to structured uncertainty and misspecification at some shock channels but not others, may be useful in interpreting results produced by more sophisticated approaches. The size of the S's and θ of can be disciplined by data driven detection probabilities as in Hansen and Sargent (2008), Chapter 9.

Turning our attention to β_2, we note that there is really no reason that β_2 needs to be the same as β. When $\beta_2 = \beta$ we see from equation 2.12 below that, if $\kappa_T = 0$, $\mathrm{E}_0 T_t$ goes to infinity as t goes to infinity, i.e., the planet becomes infinitely hot. In this case, regardless of how small $\eta_0 > 0$ is, we are still unable to find a robust control that prevents this outcome Nature draws the worst case model.

In their IA model with robustness, Hennlock (2009) and Hennlock and Sterner (2012) avoid this outcome because their dynamic temperature equation is based on the standard specification with the climate sensitivity and including the decay term $-\lambda_1 T dt$ in the temperature dynamics, and because, following the DICE formulation, they incorporate a representation of the deep ocean, which absorbs carbon from the atmosphere. Similarly, we avoid unbounded temperatures by assuming, $0 < 1 - \kappa_T < 0$.

If instead we assume $\beta_2 > \beta$, we then decrease the size of the set R2.1 relative to how future utility is discounted. In this case, the set over which we must construct a control that works reasonably well is smaller relative to discount factor β. By rewriting the Lagrangian and absorbing the ratio, $(\beta/\beta_2)^t$ into θ, we can convert the problem into the form of 2.1 where $\theta_t \equiv \theta(\beta/\beta_2)^t$ converges to zero geometrically fast.

This formulation we offer in Remark 2.1 also suggests where research should be directed when robust control fails to keep $\mathrm{E}_0 T_t$ bounded below what climate scientists consider to be a tolerable long run yearly average temperature of the planet. For example we might assume $\beta_2(f_{R\&D})$ increases for each fraction $f_{R\&D} \epsilon (0,1)$ of output $A_t \Omega_t K_t^{\alpha} E_t^{\nu}$ that is set aside for R&D that shrinks the set over which we must construct a control that works uniformly well. Of course since output could shrink on average over time a better formulation might be $\beta_2(f_{R\&D} A_t \Omega_t K_t^{\alpha} E_t^{\nu})$ instead of $\beta_2(f_{R\&D})$. We suggest exploration of this kind of modeling as an interesting topic for future research.

Finally, there remains an issue raised by Hansen and Sargent (2008) (Section 2.2, especially fn. 3). Our formulation of robust control only allows the minimizing agent to distort the conditional mean. In some applications it may be useful to allow the minimizing agent to distort the conditional variance also. (See Hansen and Sargent (2008), fn 3, page 26).

We assume full capital depreciation, $\delta = 1$, and $U(C) = \log(C)$, as well as a Cobb-Douglas production function, in 2.1 so that we can use a well-known closed form solution for consumption and capital (Sargent (1987), page 122). From the Euler equation first-order necessary conditions with respect to K_{t+1}, as a consequence of full depreciation, log utility, Cobb Douglas production, we obtain, as in Proposition 3 of

Golosov et al. (forthcoming),

$$C_t = (1-\alpha\beta)Y_t, \qquad K_{t+1} = \alpha\beta Y_t. \tag{2.3}$$

Golosov et al. (forthcoming), following standard arguments in macroeconomics (Sargent (1987), page 122) obtain the solution 2.3 under a variety of specifications of the rest of the problem. They get further towards complete closed form solutions for a version of their model by using an exponential damage function together with the Cobb Douglas specification (Golosov et al. (forthcoming), Equation (26)).

Note that the consumption to output ratio, $\frac{C_t}{Y_t} = (1-\alpha\beta)$ is constant. We will see in the computational results presented in Section 3 below that the consumption to output ratio is approximately constant in the more general model treated there.

To continue, we initially assume at first that σ_T, σ_D do not depend upon temperature and that climate does not affect growth, i.e. $\mu_{aT} = 0 = \sigma_{aT}$ in equation 2.1. Some justification for part of this assumption can be found in Matthews, Solomon and Pierrehumbert (2012) (Figure 4, page 4372), where it appears from the scatter diagram there that is constant in global average yearly temperature.

However, Moyer et al. (2013) make a strong case that climate damages can impact the rate of growth as well as the level, and this channel - which seems to have been neglected in the literature - may be more important than the usual level effects of damages. Hence, we will subsequently examine channels that impact growth rates.

Compute $\partial J/\partial E_t$ to obtain (in current value units), (after using $U'(C_{t+s}) = 1/C_{t+s} = \frac{1}{(1-\alpha\beta)Y_{t+s}}, s = 0, 1, 2, \dots$), to obtain the last line,

$$\begin{aligned} 0 = \partial J/\partial E_t = \sum_{s=0}^{\infty} \beta^{t+s} U'(C_{t+s}) \frac{\partial Y_{t+s}}{\partial E_t} - \mu_{R_0} \Rightarrow \\ U'(C_t)\frac{\partial Y_t}{\partial E_t} = -\sum_{s=1}^{\infty} \beta^s U'(C_{t+s}) \frac{\partial Y_{t+s}}{\partial E_t} + \frac{\mu_{R_0}}{\beta^t} \Rightarrow \\ \frac{\nu}{1-\alpha\beta}\frac{1}{E_t} + \frac{1}{1-\alpha\beta} \mathrm{E}_t \left\{ \sum_{s=1}^{\infty} \beta^s \left[\frac{\partial \Omega_{t+s}}{\partial T_{t+s}} \frac{1}{\Omega_{t+s}} \right] \frac{\partial T_{t+s}}{\partial E_t} \right\} = \frac{\mu_{R_0}}{\beta^t} \end{aligned} \tag{2.4}$$

where we used Eq. 2.3 to cancel out Y_{t+s} to obtain the last line of Eq. 2.4. Here μ_{R_0} denotes the Lagrange multiplier on the resource constraint at the beginning date $t = 1$. Note that since $T_t = T_0 + \lambda b(R_0 - R_t) + \hat{T}_t = T_0 + \lambda b \sum_{r=0}^{t-1} E_r + \hat{T}_t$, all terms in Eq. 2.4 with $s > 0$ are externality terms that go into the expression for the optimal carbon tax. In later work, where we include more channels of climate influence on the economy, the term $\frac{1}{\Omega_{t+s}} \frac{\partial \Omega_{t+s}}{\partial T_{t+s}}$ will be replaced by $\frac{1}{Y_{t+s}} \frac{\partial Y_{t+s}}{\partial T_{t+s}}$. In particular the term $\frac{1}{\Omega_{t+s}} \frac{\partial \Omega_{t+s}}{\partial T_{t+s}}$ will be replaced by $\frac{1}{\Omega_{t+s}} \frac{\partial \Omega_{t+s}}{\partial T_{t+s}} + \frac{1}{A_{t+s}} \frac{\partial A_{t+s}}{\partial T_{t+s}}$, when we open the A-channel as well as the Ω-channel for temperature effects.

Since $\frac{1}{A_{t+s}} \frac{\partial A_{t+s}}{\partial T_{t+s}} = -\mu_{aT}$ from Eq. 2.1 for the constant variance case, this is a simple modification that can be inserted into all of our formulae below.

We may compute optimal carbon taxes in consumption units using the last line from Eq 2.4 as follows

$$\begin{aligned} \tau_t \equiv \frac{1}{U'(C)} \left\{ \frac{1}{1-\alpha\beta} \mathrm{E}_t \left\{ \sum_{s=1}^{\infty} \beta^s \left[-\frac{1}{\Omega_{t+s}} \frac{\partial \Omega_{t+s}}{\partial T_{t+s}} \right] \frac{\partial T_{t+s}}{\partial E_t} \right\} \right\} \\ = Y_t \mathrm{E}_t \left\{ \sum_{s=1}^{\infty} \beta^s \left[-\frac{1}{\Omega_{t+s}} \frac{\partial \Omega_{t+s}}{\partial T_{t+s}} \right] \frac{\partial T_{t+s}}{\partial E_t} \right\}. \end{aligned} \tag{2.5}$$

Here Greek E_t is conditional expectation. The quantity τ_i is the marginal social cost in units of consumption caused by using an additional unit of energy. This is the Pigou energy tax imposed on competitive energy using firms needed to implement the social optimum in a competitive equilibrium

The partial derivative, $\frac{\partial T_{t+s}}{\partial E_t} = \lambda b$, is a simple expression. The expression for $\frac{1}{\Omega_{t+s}} \frac{\partial \Omega_{t+s}}{\partial T_{t+s}}$ can be quite complicated depending upon the specification of the Ω dynamics, i.e. if we specify something more complicated than Eq. 2.2. We stick to the case specified in Eq. 2.2, i.e. $D_t(T_t) = D_t(T_t - T_1)$ but it is important to note that one can easily extend our formulas to functions of the form $D_t(T_t) = D_t L(T_t - T_1)$ where $L(z)$ is a convex increasing function with $L^{'}(z) > 0, L^{''}(z) \geq 0$. This kind of extension is useful in order to capture commonly expressed concerns that temperature increases beyond $(T_t - T_0) > 2$ degrees Centigrade might lead to dangerous climate change which could seriously impact production. In these two cases we obtain

$$\frac{1}{\Omega_{t+s}}\frac{\partial \Omega_{t+s}}{\partial T_{t+s}} = -D_{t+s}$$
$$\frac{1}{\Omega_{t+s}}\frac{\partial \Omega_{t+s}}{\partial T_{t+s}} = -D_{t+s}L^{'}(T_{t+s}-T_0). \tag{2.6}$$

For the particular case, $L(z) = z$, Eq. 2.4 simplifies to the following

$$\frac{\nu}{(1-\alpha\beta)}\frac{1}{E_t} + \frac{\lambda b}{1-\alpha\beta}\mathrm{E_t}\left\{\sum_{s=1}^{\infty}\beta^s(-D_{t+s})\right\} = \frac{\mu_{R_0}}{\beta^t} \tag{2.7}$$

$$E_t = \frac{\beta^t}{[\mu_{R_0}(1-\alpha\beta)/\nu] + \beta^t(\lambda b/\nu)\Delta_t}$$
$$\Delta_t \equiv \mathrm{E}_t\left\{\sum_{s=1}^{\infty}\beta^s D_{t+s}\right\} \tag{2.8}$$

$$\tau_t = \lambda b Y_t \mathrm{E}_t\left\{\sum_{s=1}^{\infty}\beta^s D_{t+s}\right\} = \lambda b Y_t \Delta_t. \tag{2.9}$$

Note that this formula is also quite straightforward. The term $\beta^t(\lambda b/\nu)\Delta_t$ in the denominator of Eq. 2.8 is positive and captures the negative spillover effects of fossil fuel usage on how much energy is used at each point in time. Since it is positive and in the denominator, energy usage will be smaller at each point in time. Since $\beta < 1$, this effect fades away as t goes to infinity. If it is zero, Eq. 2.7 is related to the discrete time Hotelling formula for energy usage in the absence of externalities.

Note that for the case $D_{t+s} = d, \tau_t = \lambda b Y_t \frac{\beta}{1-\beta}$ the carbon tax is rising (falling) over time when $\tau_{t+1}/\tau_t = Y_{t+1}/Y_t > 1$, ($\tau_{t+1}/\tau_t = Y_{t+1}/Y_t < 1$) i.e. when the economy is growing (shrinking).

If we open up the channel for temperature to impact the growth rate of the economy we simply add the term μ_{aT} to D_{t+s} at each date $t+s$ in formula 2.9. We see that the optimal tax on carbon increases (as a fraction of output), as we would expect from Moyer et al. (2013). If there is also the robustness channel, $\sqrt{\epsilon}\sigma_{aT}(T_{t+s}-T_0)(-G_{aT,t+s}+e_{aT,t+s+1})$ added then yet another term will be added to formula 2.9 to account for the marginal impact of E_t through temperature, T_{t+s} upon this channel.

It is important to note that for the more general case in Eq. 2.6 where D_{t+s} is replaced by $D_{t+s}L^{'}(T_{t+s}-T_0)$, that the carbon tax may be quite large if $L^{'}(T_{t+s}-T_0)$ increases quite rapidly if $(T_{t+s}-T_0)$ becomes greater than 2 degrees Centigrade and the science is right that temperature increases of more than 2 degrees Centigrade for world average yearly temperature is getting in to such serious climate change that world production is seriously hampered. This is why it may be important to keep in mind the case where $L(z)$ is an increasing convex *nonlinear* function.

Notice, however, that the unit carbon tax itself may *decrease* as thetas increase where we use the size of a channel's theta as a measure of robustness concern arising from that channel. That is, contrary to what might be initially supposed, opening robustness channels may *decrease* the unit carbon tax. To put it another way, there are two opposing forces. Increasing robustness, e.g., by increasing thetas, causes damage to output. This force causes the unit tax rate to fall because there is a smaller level of output to damage when thetas increase. In order for increased thetas to decrease the unit carbon tax rate, the ratio of the tax rate to output must fall by enough to cancel the decrease in the level of output. It may seem odd on first thought that the tax per unit of E could fall when some robustness channels are opened up. But it makes sense that the tax per unit of E should be higher the higher is the size of the economy, because there is more economy to damage when using an extra unit of E raises the temperature. Hence if opening a robustness channel causes the size of the economy to fall under the worst case model, it makes sense that the tax per unit E might fall because now there is less economy to damage.

Later on we will assume, in some cases, Δ_t is constant over time. Compare our equation 2.9 to Golosov et al. (forthcoming) (equation (13), under their assumption A.2). Note that we avoid their concern with climate sensitivity of Roe and Baker by using the Matthews et al. (2009) cumulative climate response (CCR) formulation. This allows us to use a simpler carbon cycle than the three-layer DICE formulation (Nordhaus (2008)) that Golosov et al. (forthcoming) adopt, as well as to avoid issues arising from the fat-tailed uncertainty of the climate sensitivity parameter (Roe and Baker (2007), Roe (2013*b*)).

Note that τ_t is the marginal damage cost in units of consumption and corresponds to Λ_t^s in equation (11) of Golosov et al. (forthcoming). But in our case, because of the CCR, we may not obtain the Golosov et al. (forthcoming) and Brock, Engstrom and Xepapadeas (forthcoming) result that marginal damage costs, and hence Pigou tax per unit energy usage, may fall over time. In our model the Pigou tax is most likely to rise over time since $\{Y_t\}$ is likely to rise over time. However, in the worst-case model, it is possible that $\{Y_t\}$ may fall over time. We say more about this below when we turn to a detailed treatment of robustness. This will be especially the case when we open up the channel in which degraded climate may impact the *rate* of growth as well as levels through the Ω function.

Golosov et al. (forthcoming) discuss cases where the marginal damages from using a unit of fossil fuels are large enough that it will not be optimal to use all the fossil fuels in the ground, so in that case, μ_{R_0} turns out to be zero.

We turn now to the robustness G's where $\beta_2 = \beta$, so that the S's can be absorbed into theta to produce θ_i's for $i \in \{a, D, aT, c\}$ that are constant in time t. It is important to note that if $\kappa_T = 0$ in the dynamics of the fluctuation-about-trend term, $\hat{T}_t$, then under robustness, it is possible that $\mathrm{E}_0 \hat{T}_t \to \infty, t \to \infty$, i.e. we lose control of the temperature unless we put in, at least, a small decay term, $\kappa_T > 0, 0 < 1 - \kappa_T < 1$. We believe that it is reasonable to do so, because the CCR approach of Matthews et al. (2009), Matthews, Solomon and Pierrehumbert (2012) is intended for use at medium range time scales.

We begin with the calculation of G_{at} closing down the aT channel in Eq. 2.1 in order to obtain the simplest formulas 2.10 for G_{at} and for the distortion to the A-dynamics below:

$$
\begin{gathered}
0 = \frac{\partial J}{\partial G_{at}} = \beta^t \left\{ \beta G_{at}/\theta_a + \sum_{s=1}^{\infty} \beta^{s-1} U^{'}(C_{t+s}) \frac{\partial Y_{t+s}}{\partial G_{at}} \right\} \Rightarrow \\
G_{at} = \theta_a (\frac{1}{1-\alpha\beta}) \mathrm{E}_t \left\{ \sum_{s=1}^{\infty} \beta^s \frac{1}{A_{t+s}} \frac{\partial A_{t+s}}{\partial G_{at}} \right\} = \frac{\theta_a \sqrt{\epsilon}\sigma_a}{(1-\alpha\beta)(1-\beta)} \\
\log A_{t+1} - \log A_t = \mu_{a0} - \frac{\epsilon \sigma_a^2 \theta_a}{(1-\alpha\beta)(1-\beta)} + \sqrt{\epsilon}\sigma_a e_{a,t+1}.
\end{gathered} \tag{2.10}
$$

The calculations of the other G's proceed similarly. E.g., we obtain, (provided that all other robustness channels that can be impacted by temperature changes are closed down),

$$
\begin{aligned}
0 = \frac{\partial J}{\partial G_{ct}} &= \beta^t \left\{ \beta G_{ct}/\theta_c + \sum_{s=1}^{\infty} \beta^s U'(C_{t+s}) \frac{\partial Y_{t+s}}{\partial G_{ct}} \right\} \Rightarrow \\
\frac{\beta}{\theta_c} G_{ct} &= -\left(\frac{1}{1-\alpha\beta}\right) \mathrm{E}_t \left\{ \sum_{s=1}^{\infty} \beta^s \frac{1}{\Omega_{t+s}} \frac{\partial \Omega_{t+s}}{\partial G_{ct}} \right\} \\
&= -\frac{1}{(1-\alpha\beta)} \beta \mathrm{E}_t \left\{ \sum_{s=1}^{\infty} \beta^{s-1} \left[-D_{t+s} \frac{\partial \hat{T}_{t+s}}{\partial G_{ct}} \right] \right\} \\
&= -\frac{\sqrt{\epsilon}\sigma_T}{1-\alpha\beta} \beta \mathrm{E}_t \left\{ \sum_{s=1}^{\infty} (\beta(1-\kappa_T))^{s-1} D_{t+s} \right\} \\
&= -\frac{\left[\frac{\sqrt{\epsilon}\sigma_T}{1-\alpha\beta}\beta d\right]}{1-\beta(1-\kappa_T)}
\end{aligned} \tag{2.11}
$$

where the last line holds for the special case, $D_{t+s} = d, \forall\{t,s\}$. Equation 2.11 allows us to see how robustness creates a distortion in the temperature dynamics, as seen in Equation 2.12:

$$
\begin{aligned}
T_{t+1} - T_t &= \lambda b E_t + \hat{T}_{t+1} - \hat{T}_t \\
\hat{T}_{t+1} - \hat{T}_t &= -\kappa_T \hat{T}_t + \frac{\epsilon\theta_c\sigma_c^2}{1-\alpha\beta} \mathrm{E}_t \left\{ \sum_{s=1}^{\infty} (\beta(1-\kappa_T))^{s-1} D_{t+s} \right\} + \sqrt{\epsilon}\sigma_T e_{T,t+1}.
\end{aligned} \tag{2.12}
$$

We emphasize that formula 2.12 is derived under the assumption that all other robustness channels that can be impacted by temperature are closed down. If we open a robustness channel in the damage function, i.e. add the term $\sqrt{\epsilon}\sigma_D(-G_{Dt} + e_{D,t+1})$ to the exponent in Ω_{t+1} we can compute the distortion from this channel as follows,

$$
\begin{aligned}
0 = \frac{\partial J}{\partial G_{Dt}} &= \beta^t \left\{ \beta G_{Dt}/\theta_D + \sum_{s=1}^{\infty} \beta^s U'(C_{t+s}) \frac{\partial Y_{t+s}}{\partial G_{Dt}} \right\} \\
\Omega_{t+1} &= \exp\left[-D_{t+1}(T_{t+1} - T_0) - \frac{\theta_D \epsilon \sigma_D^2}{1-\alpha\beta} + \sqrt{\epsilon}\sigma_D e_{D,t+1} \right].
\end{aligned} \tag{2.13}
$$

The simplicity of 2.13 results from the fact that the extra robustness term, $\sqrt{\epsilon}\sigma_D(-G_{Dt} + e_{D,t+1})$ does not appear in the temperature dynamics. Any robustness term that appears in the temperature dynamics sets off effects that appear in every future date. These last several equations illustrate how this simple closed form example leads to insights about the results of a robust control analysis in this particular context. For example if

$$\mu_{a0} - \frac{\epsilon \sigma_a^2 \theta_a}{(1-\alpha\beta)(1-\beta)} < 0 \tag{2.14}$$

we see that the level of technology decreases over time, i.e., negative growth occurs under the worst case model, even if the A-channel is shut off. Of course if ϵ is small enough, we should be far from "breakdown" values of parameters that would imply that 2.14 holds. We hasten to point out that 2.14 could be considered implausible - it represents the level of technology stochastically decreasing in the absence of a channel through which climate itself impacts the rate of growth, as in Moyer et al. (2013). We return to this point later. We note, however if β is close to one, which it is, then 2.14 may well hold. But we emphasize that if the A-channel for climate degradation to affect growth is opened up then, under the worst case model, the possibility of negative growth is increased, as we would expect from Moyer et al. (2013).

Remark 2.2 (**Directions of Directed Technological Change**). *Our ignorance about specification of technological dynamics, which needs to be captured in the size of the S's and the size of θ, may reside more in forecasting the* direction *of "directed technological change" which may severely affect the environment and the climate. In particular if negative externalities are not "fully taxed" at marginal social cost, directed technological change may occur to exploit these "environmental subsidies". Potential examples of environmentally harmful directed technological change are fracking, tar oil sands, large scale commercial fishing technologies, etc. Dealing with this requires a more detailed model than our current model. A simple way of doing it might be to insert a term depending upon A_i into the damage equation. But what we really need to model is the incentive to direct technological change towards exploitation of privately free but socially costly goods such as un-priced environmental goods, e.g. the quality of the atmosphere. Uncertainties about whether human societies will be able to design collective institutions to control these kinds of mal-incentives may be the biggest uncertainties of all. Nordhaus has commented on this issue in various places.*

We turn next to the distorted temperature dynamics equation 2.12. Equation 2.12 implies that if we set energy use equal to zero, i.e., $E_s = 0$ for all dates s greater than a fixed date t, and if $\kappa_T = 0$, then the temperature will still increase stochastically to infinity under the worst case model. This is implausible because it overstates any conceivable long-run "climate commitment" from economic activity; i.e., the earth?s surface temperature cannot increase without bound. However, given our objective of only using the Section 2 model to help interpret the computational results in Section 3 and to help understand "vulnerability points" of sets of mis-specifications around a baseline model that a robust planner must protect against, this result is useful. For example it shows us where resources need to be directed in order to improve specification of the dynamics. That is, this implausibility suggests changing the scaling of either

θ_c or ϵ to decrease fast enough with time s, or to change the specification of the damage dynamics equation and/or the temperature dynamics equation. We discussed some routes to get around this type of result in Remark 2.1 above. Turn now to introducing a robustness channel that reflects uncertainty in the size of the key CCR parameter, λ.

2.1 Robustness Channel Reflecting Uncertainties in CCR

Matthews et al. (2009), Matthews, Solomon and Pierrehumbert (2012) argue that the CCR, denoted by λ in this paper, is approximately constant. However the Figures in their papers show quite a wide range of uncertainty of values of CCR across models in the C4MIP simulations as well as what they called "observational estimates of CCR" (e.g. Matthews et al. (2009), Figure 3 for the C4MIP simulations and Figure 4 for "observational estimates of CCR"). Notice also that even though the CCR for each model eventually settles down to a constant value for each model in the C4MIP simulations, there is first a period of "transient" behavior (Matthews et al. (2009), Figure 3b). We capture this uncertainty by the following discrete time Ornstein-Uhlenbeck process with robustness channel, $G_{\lambda,t}$ added,

$$\lambda_{t+1} - \lambda_t = (a_\lambda - b_\lambda \lambda_t) + \sqrt{\epsilon}\sigma_\lambda(\tilde{e}_{\lambda,t+1} - G_{\lambda,t}) \tag{2.15}$$

where

$$\frac{a_\lambda}{b_\lambda} = \overline{\lambda}, \tag{2.16}$$

the parameters can be chosen to very roughly match the Figures in Matthews et al. (2009), Figures 3,4, and the central value of λ , denoted by $\overline{\lambda}$ can be chosen to match the central value shown in these figures. If $G_{\lambda,t}$ is fixed at zero, if $0 < b_\lambda < 1$, if $\tilde{e}_{\lambda,t}$ is IIDN(0,1), then process 2.15 converges to a Gaussian process with mean and variance,

$$\sigma^2_{\lambda,\infty}(\epsilon) = \frac{\epsilon\sigma^2_\lambda}{1-(1-b_\lambda)^2}. \tag{2.17}$$

We shut off all other channels of robustness in the temperature equation in order to focus on what robustness in λ adds. The λ-robustness channel adds another term to the temperature dynamics as follows,

$$\begin{aligned} T_{t+1} &= T_0 + \lambda_{t+1} b(R_0 - R_{t+1}) + \hat{T}_{t+1} \\ &= T_0 + \left[a_\lambda + (1-b_\lambda)\lambda_t + \sqrt{\epsilon}\sigma_\lambda(-G_{\lambda,t} + \tilde{e}_{\lambda,t+1})b\right](R_0 - R_{t+1}) + \hat{T}_{t+1} \end{aligned} \tag{2.18}$$

where we add the cost terms, $\frac{\beta}{2\theta_\lambda} G^2_{\lambda,t}$ to the objective function in 2.1. Using the damage function $\Omega_{t+s} = \exp\left[-d(T_{t+s} - T_0)\right]$ for simplicity and differentiating the objective function in 2.1 w.r.t. $G_{\lambda,t}$ gives us the following,

$$\begin{aligned} 0 = \frac{\partial J}{\partial G_{\lambda t}} &= \beta^t \left(\beta G_{\lambda t}/\theta_\lambda\right) + \sum_{s=1}^{\infty} \beta^{t+s} U^{'}(C_{t+s}) \frac{\partial Y_{t+s}}{\partial G_{\lambda t}} \Rightarrow \\ G_{\lambda t} &= -\theta_\lambda \sum_{s=1}^{\infty} \beta^{s-1} U^{'}(C_{t+s}) \left(-dY_{t+s} \frac{\partial T_{t+s}}{\partial G_{\lambda t}}\right) \\ = \frac{d\theta_\lambda}{(1-\alpha\beta)} \sum_{s=1}^{\infty} \beta^{s-1} \frac{\partial T_{t+s}}{\partial G_{\lambda t}} &= \frac{bd\theta_\lambda}{(1-\alpha\beta)} \sum_{s=1}^{\infty} \beta^{s-1} (R_0 - R_{t+s}) \frac{\partial \lambda_{t+s}}{\partial G_{\lambda t}} \\ &= -\frac{\sqrt{\epsilon}\sigma_\lambda\theta_\lambda bd}{(1-\alpha\beta)} \sum_{s=1}^{\infty} \left(\beta(1-\beta_\lambda)\right)^{s-1} (R_0 - R_{t+s}) \end{aligned} \tag{2.19}$$

where the last lines follow from log utility, Cobb Douglas production function, and full depreciation of capital and 2.18. From 2.19 we have the following temperature dynamics,

$$\begin{aligned} T_{t+1} &= T_0 + \lambda_{t+1} b(R_0 - R_{t+1}) + \hat{T}_{t+1} \\ &= T_0 + \left[a_\lambda + (1-b_\lambda)\lambda_t + \sqrt{\epsilon}\sigma_\lambda(-G_{\lambda,t} + \tilde{e}_{\lambda,t+1})\right] b(R_0 - R_{t+1}) + \hat{T}_{t+1}. \end{aligned} \tag{2.20}$$

Since $R_0 - R_{t+1} = \sum_{r=0}^{t} E_r$ is cumulative emissions at date t, we see from 2.19 that the distortion could be quite large depending upon the size of parameters, e.g. if the persistence parameter b_λ is small the sum in 2.19 would be quite large, especially if cumulative emissions, $R_0 - R_{t+s}$ increase substantially over time. This finding argues for more research effort to be allocated towards reducing uncertainty in knowledge of the value of the CCR parameter, λ. Let us now compute the FONC for E_t. Differentiating J with respect to E_t, we obtain

$$0 = \frac{\partial J}{\partial E_t} = -\mu_{R_0} + \beta^t U^{'}(C_t)\frac{\partial Y_t}{\partial E_t} + \sum_{s=1}^{\infty} \beta^s U^{'}(C_{t+s})\frac{\partial Y_{t+s}}{\partial E_t} \Rightarrow$$

$$\begin{aligned} \frac{\partial Y_t}{\partial E_t} = \frac{1}{U'(C_t)}\frac{\mu_{R_0}}{\beta^t} - \frac{1}{U'(C_t)}\sum_{s=1}^{\infty} \beta^s U^{'}(C_{t+s})\frac{\partial Y_{t+s}}{\partial E_t} \\ = \frac{1}{U'(C_t)}\frac{\mu_{R_0}}{\beta^t} + \tau_t \end{aligned} \tag{2.21}$$

where the first term in the last line is the "Hotelling" price of the resource in consumption units and the second term is the Pigou externality tax per unit of E measured in consumption units, i.e.

$$\tau_t = -\frac{1}{U'(C_t)}\sum_{s=1}^{\infty} \beta^s U^{'}(C_{t+s})\frac{\partial Y_{t+s}}{\partial E_t}. \tag{2.22}$$

We compute an explicit formula for the externality tax. We have,

$$\begin{aligned} \tau_t = -\frac{1}{U'(C_t)}\sum_{s=1}^{\infty} \beta^s U^{'}(C_{t+s})\frac{\partial Y_{t+s}}{\partial E_t} \\ = Y_t d \sum_{s=1}^{\infty} \beta^s \frac{\partial T_{t+s}}{\partial E_t} = Y_t d \sum_{s=1}^{\infty} \beta^s b \lambda_{t+s}. \end{aligned} \tag{2.23}$$

What do we learn about the size and dynamics of the carbon tax τ_t and the dynamics of the CCR parameter λ_t from this exercise? Since we have seen from 2.19 that the distortion to the λ dynamics can be quite large, i.e. the CCR parameter could become quite large, under the "worst case" model with modest use of fossil fuels which, in turn leads to large increases in temperature and, hence, large damages under the worst case model. We believe that this reasoning plus the potential usefulness of the CCR parameter in formulating carbon budgets argues for a strong effort to reduce our ignorance (embodied in the size of θ_λ) about the "true" size of the CCR parameter λ.

At this point in the paper we digress to discuss why it is important to study the impact of robustness and uncertainty on the CCR parameter, λ and to direct research towards reducing our ignorance about the "true" size of λ and its dynamics. Matthews, Solomon and Pierrehumbert (2012) argue that the

CCR framework, i.e. focusing on cumulative carbon budgets that avoid dangerous climate change with high probability is a good policy framework for achieving climate stabilization (Matthews, Solomon and Pierrehumbert (2012), page 4365). The Fifth Assessment Report of the IPCC has very recently argued for a carbon budgeting approach.

We quote extensively from Lord Stern where he talks about the IPCC report and carbon budgeting here,

> This is a comprehensive and important report, prepared by 259 experts from 39 countries, and governments and other policy-makers should read it very carefully. The report makes clear that the Earth is warming and the climate is changing, that human activities are primarily responsible, and that without very strong cuts in emissions of greenhouse gases, we face huge risks from global warming of more than 2°C by the end of this century compared with the period before the Industrial Revolution. All governments have already agreed that it would be dangerous to exceed a threshold of global warming by 2°C. Delay is dangerous because greenhouse gases are accumulating in the atmosphere and because we are locking in high-carbon infrastructure and capital.
>
> This report points out that to have at least a 50 per cent chance of keeping within that limit, we must emit in total no more than about 820 and 1445 billion tonnes of carbon dioxide and other greenhouse gases during the rest of this century. Given that the world is currently emitting about 50 billion tonnes of greenhouse gases in terms of carbon-dioxide-equivalent each year, this report implies that, even if we were to stay at current levels, we would exhaust the emissions budget within 15 to 25 years. And if we continue to increase annual emissions, the budget will be depleted even sooner. That is why I think nations, cities, communities and companies will recognize the importance of these findings and will increase the urgency and scale of the emissions reductions that they are planning to undertake. I also expect this emissions budget to focus the minds of governments in the international negotiations towards a new climate change treaty, to be signed in Paris at the end of 2015. The transition to a low-carbon economy, led by private sector investment, in the context of sound public policy, will be full of opportunity, discovery, innovation and growth. (Stern (2013*a*))

Given the recent importance key policymakers are placing on cumulative carbon budgeting, e.g. Lord Stern's quote above, therefore we believe that it is of key importance to study robustness in the CCR framework and, not only that, but also consider explicitly a robustness channel on the key CCR parameter λ. This is so because once it is agreed by policy makers that the threshold of 2 degrees Centigrade is the threshold for "dangerous climate change" then the size of the recommended cumulative carbon budget will depend upon the CCR parameter, λ as well as the dynamics of the stochastic process driving shocks to the temperature dynamics as well as any robustness considerations.

Suppose policymakers know the "true value" of the CCR parameter, λ, and the only uncertainty is in shocks to temperature as in 2.12 and $D_{t+1+r} = d > 0$, $\forall t, r$. We recall this version of 2.12 here for ease of reading,

$$T_{t+1} - T_t = \lambda b E_t + \hat{T}_{t+1} - \hat{T}_t$$
$$\hat{T}_{t+1} - \hat{T}_t = -\kappa_T \hat{T}_t + \frac{\epsilon \theta_c \sigma_c^2}{1 - \alpha\beta} / [1 - \beta(1 - \kappa_T)] + \sqrt{\epsilon} \sigma_T e_{c,t+1}. \tag{2.24}$$

Since policymakers are assumed to be fully confident in their model, the robustness parameter, $\theta_c = 0$. In order to keep things as simple as possible, we assume $\kappa_T = 0$ so that $\hat{T}_t = 0$ for all dates t. If there are no shocks to temperature so that $\sigma_c = 0$, we may sum 2.24 obtain the cumulative carbon budget that keeps $T_t - T_0 \leq 2°\text{C}$,

$$2°\text{C} \geq T_t - T_0 = \lambda \left\{ \sum_{s=1}^{t-1} b E_s \right\}, i.e.,$$
$$\sum_{s=1}^{t-1} b E_s \leq \frac{2°\text{C}}{\lambda}, \quad t = 0, 1, 2, \ldots \tag{2.25}$$

If now there are shocks to temperature, i.e., $\sigma_T > 0$, the same type of argument yields the precise version of Lord Stern's statement made above,

$$Pr \left\{ \sum_{s=0}^{t-1} b E_s \leq \left[\frac{2°\text{C} - \sqrt{\epsilon} \sigma_T \sum_{s=0}^{t-1} \tilde{e}_{T,s+1}}{\lambda} \right] \right\} \leq 1/2, \qquad t = 0, 1, 2, \ldots \tag{2.26}$$

If we assume the stochastic process, $\tilde{e}_{Tt}$, is Independent and Identically Distributed Normal with mean zero and variance one (IIDN(0,1)), then using properties of the normal distribution we may translate 2.26 to

$$F_{N(0,1)} \left\{ \left[\frac{2°\text{C} - \lambda b \sum_{s=1}^{t-1} E_s}{t \sqrt{\epsilon} \sigma_T} \right] \right\} \geq 1/2, \quad t = 0, 1, 2, \ldots \tag{2.27}$$

where $F_{N(0,1)}(x)$ is the cumulative distribution function of the standardized normal distribution. We see from 2.27 that we basically have to budget carbon to achieve the deterministic goal in 2.25 in order to achieve the desired objective 2.26. Of course if we add a slight decay term to the temperature dynamics or we replace the IID stochastic process with, for example, a mean reverting process, then the objective 2.26 may be easier to achieve by carbon budgeting than the deterministic objective 2.25.

We believe the basic take home point of robustness arguments on the use of the CCR approach is to do cumulative carbon budgeting under the assumption that the CCR parameter may be in the range of larger values depicted in Figures 3 and 4 of Matthews et al. (2009)(page 831).

These results are directly relevant to the point made by Roe (2013*b*) on the relative roles of economic and moral principles in the formulation of climate policy given deep and persistent uncertainty in climate science and economics. He argues that this uncertainty - in effect, the fundamental limits to our knowledge in this context - elevate the appropriate role for moral reasoning in addressing the climate problem. From this perspective, the use of the CCR and the bounds derived in equations 2.25-2.27 can be seen as embodying the "cautionary" and arguably morality-based criterion of limiting global mean temperature to a 2-degree increase. Applying this decision rule would approximately correspond to using any of a range of damage functions that are "very steep" above the 2-degree threshold, an appealing heuristic given continuing deep uncertainty regarding the appropriate specification of damage functions Pindyck (2013*b*).

3 Numerical illustrations

This section provides solutions to stochastic and deterministic finite horizon robust and non-robust problems. Section 3.1 describes the finite horizon problem, Section 3.2 provides example deterministic solutions and Section 3.3 discusses a stochastic example.

Although it its tempting to think that the introduction of robustness into integrated models of the environment necessarily will entail a more conservative usage of energy and entail optimal policies that mitigate the impact of fossil fuels on the global climate, we show that this is not always the case. Robust policies can lead to less, the same, or more energy usage depending upon agent's preferences and the specification of the economy. Robust optimal tax rates can be lower, the same, or higher than non-robust optimal rates. We give examples of more intensive and less intensive optimal energy usage, and higher or lower tax rates in Section 3.2.

3.1 Finite horizon stochastic robust problem

We focus on a version of the general model discussed earlier with a finite horizon. Preferences are:

$$\sum_{t=0}^{S-1}\left[\beta^t \frac{C_t^{1-\gamma}}{1-\gamma} + \frac{\beta}{2\theta}\left(G_{ct}^2 + G_{at}^2 + G_{dt}^2\right)\right] + \beta^S W\left(K_S, R_S, \hat{T}_S, \log A_S, \log D_S\right) \tag{6a}$$

and the constraints are for $t = 0, 1, \ldots S-1$

$$K_{t+1} = Y_t + (1-\delta)K_t - C_t \tag{6b}$$

$$R_{t+1} = R_t - E_t \tag{6c}$$

$$\hat{T}_{t+1} = (1-\kappa_T)\hat{T}_t + \sqrt{\epsilon}\sigma_T\left(e_{T,t+1} - G_{ct}\right) \tag{6d}$$

$$\log A_{t+1} = \log A_t + \mu_{a0} - \mu_{aT}\left(T_t - \dot{T}\right) + \sqrt{\epsilon}\sigma_a\left(e_{a,t+1} - G_{at}\right) \tag{6e}$$

$$\log D_{t+1} = (1-\kappa_d)\log D_t + \kappa_d\mu_d + \sqrt{\epsilon}\sigma_d\ \left(e_{d,t+1} - G_{dt}\right) \tag{6f}$$

where

$$T_t = \bar{T}_0 + \lambda\left(R_0 - R_t\right) + \hat{T}_t \tag{6g}$$

$$Y_t = A_t\Omega_t K_t^{\alpha} E_t^{\nu} \tag{6h}$$

$$\Omega_t = \exp\left[-D_t\left(T_t - \dot{T}\right)\right] \tag{6i}$$

$$C_t, K_{t+1}, E_t, R_{t+1} \geq 0 \tag{6j}$$

and where $K_0, R_0, \bar{T}_0, \hat{T}_0, A_0$ and D_0 are given. The random variables $e_{c,t+1}, e_{a,t+1}$ and $e_{d,t+1}$ are i.i.d. standard normal random variables that are uncorrelated with each other. The agent wants to maximize the expected value of 6a by choice of adaptive process for consumption and energy and minimize it by choice of adaptive process for G_{at}, G_{ct} and G_{dt} subject to the constraints 6b through 6j.

The terminal value function is

$$W(K, R, \hat{T}, \log A, \log D) = \frac{C^{1-\gamma}}{1-\gamma}$$

where

$$\begin{aligned}
&C = Y + (1-\delta)K \\
&E = R \\
&T = \bar{T}_0 + \lambda\left(R_0 - R\right) + \hat{T} \\
&Y = A\Omega K^{\alpha} E^{\nu} \\
&\Omega = \exp\left[-\mu_d\left(T - \dot{T}\right)\right].
\end{aligned}$$

The terminal value function assumes all remaining energy is immediately used in production and all remaining capital is immediately consumed. The damages coefficient is assumed to be μ_d, regardless of the current value of D.

As compared to the problem presented in line 2.1, we set $\theta_{aT} = 0$ and the rest of the θ's to a common value so that $\theta \equiv \theta_a = \theta_c = \theta_D$. We also set $b = 1$ and take $G_{aT,t} = 0$ in all examples. In addition we represent the baseline temperature with $\dot{T}$ rather than T_0.

3.2 Examples of deterministic solutions

In this section we provide numerical solutions for deterministic robust and non-robust problems. In the non-robust deterministic planner's problem $\epsilon = 0$ and $\theta = 0$. Consequently the optimal choices of G_{ct}, G_{at} and G_{dt} are zero at every date and all the terms involving G_{ct}, G_{at} and G_{dt} can be removed from the problem. In the robust deterministic problem, we replace the robustness parameter, θ, with θ/ϵ and look at the limiting dynamic game as $\epsilon \downarrow 0$.

In Section 3.2.1, we give an example where $\gamma > 1$ and show that energy usage decreases as robustness increases. In Section 3.2.2, we give an example where $\gamma < 1$ and show that energy usage is increasing as robustness increases. In both numerical examples we set $\beta = 0.95$, the parameters directly related to temperature and environmental damages as:

$$\lambda = 0.01 \qquad \kappa_T = 0.01 \qquad \sigma_T = 0.01 \qquad \kappa_d = 0.01 \qquad \mu_d = \log 0.01 \qquad \sigma_d = 0.2 \qquad \dot{T} = 13,$$

and other parameters which primarily determine productivity, output and capital as

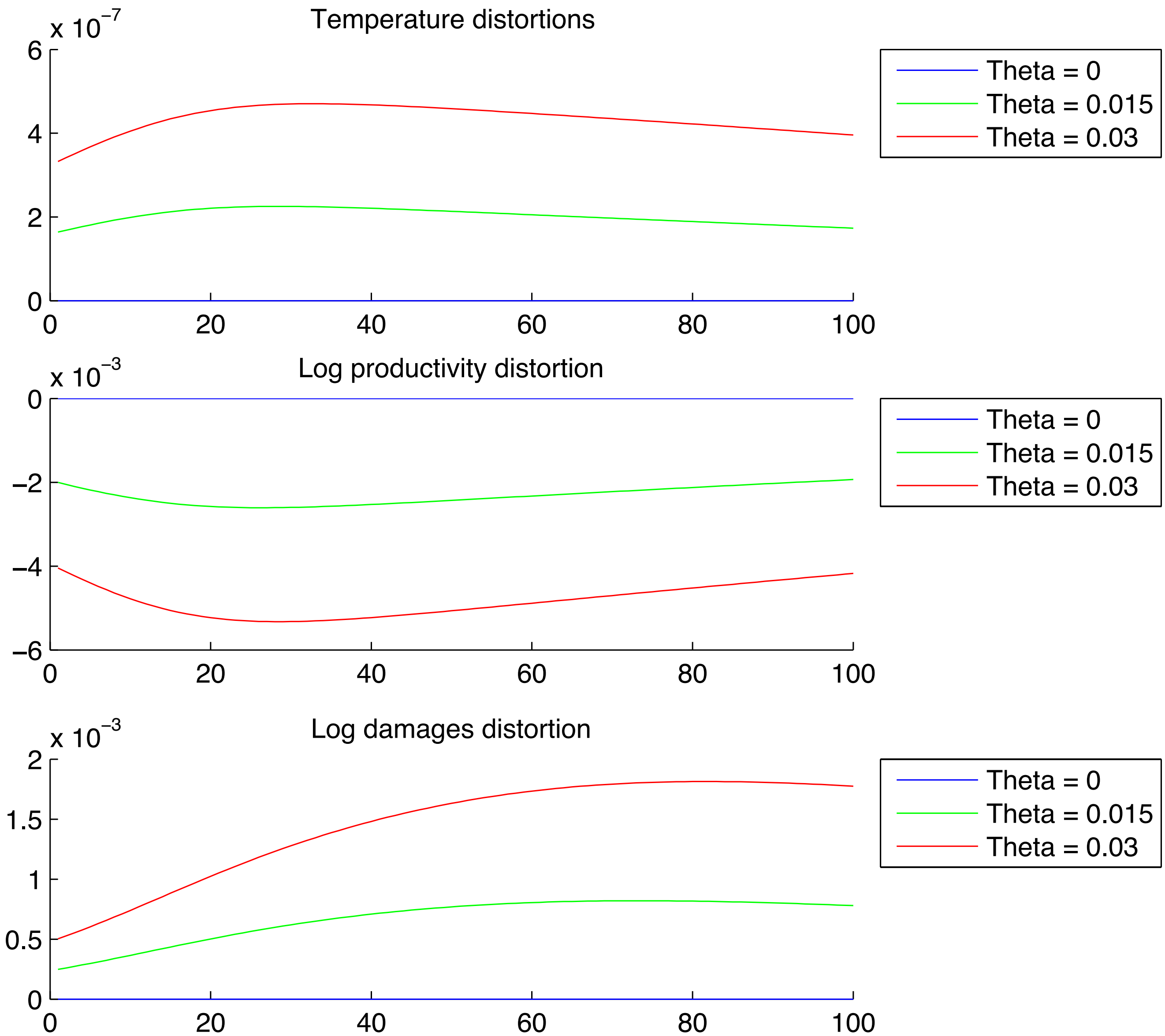
$$\mu_{a0} = 0.02 \qquad \mu_{aT} = 0 \qquad \sigma_a = 0.1 \qquad \alpha = 0.3 \qquad \nu = 0.03 \qquad \delta = 0.04.$$

The initial conditions are:

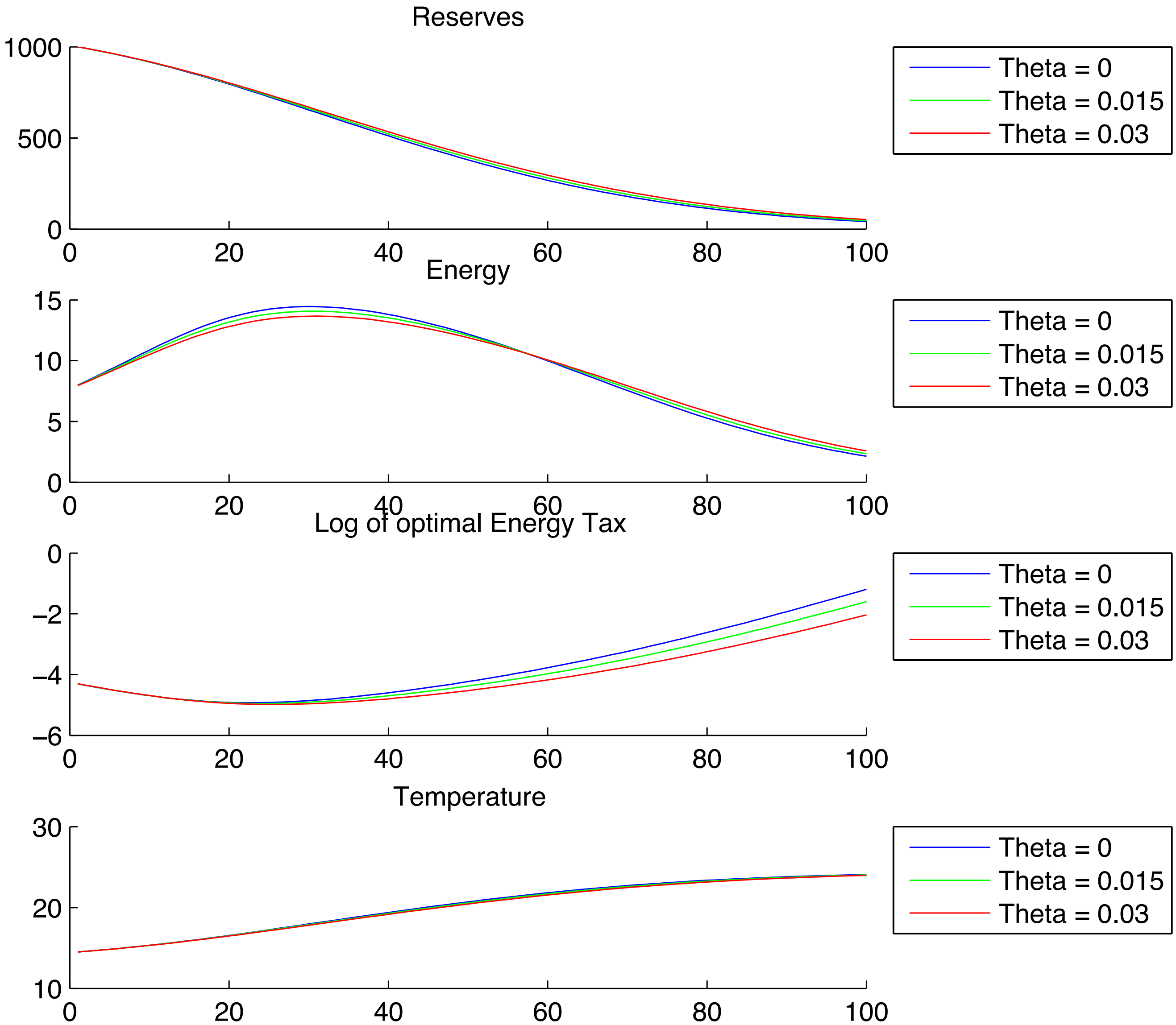
$$R_0 = 1000 \qquad \bar{T}_0 = 14.5 \qquad \hat{T}_0 = 0.0 \qquad K_0 = 60 \qquad A_0 = 1.$$

We set the number of periods to $S = 600$ in the finite horizon approximations, though we only plot the first 100 years of the solution. Because of the large value of S, the solutions for the first few years are nearly identical to the solution of the corresponding infinite horizon problem. In the graphs, solutions labeled $\theta = 0$ refer to solutions to non-robust deterministic problems. Solutions labeled with a θ greater than zero are solutions to the limiting dynamic game where the robustness parameter is θ/ϵ and $\epsilon \downarrow 0$. When $\theta > 0$, the solutions provide the evolution of state variables under the worst case model that the robust agent worries about, not the reference model.

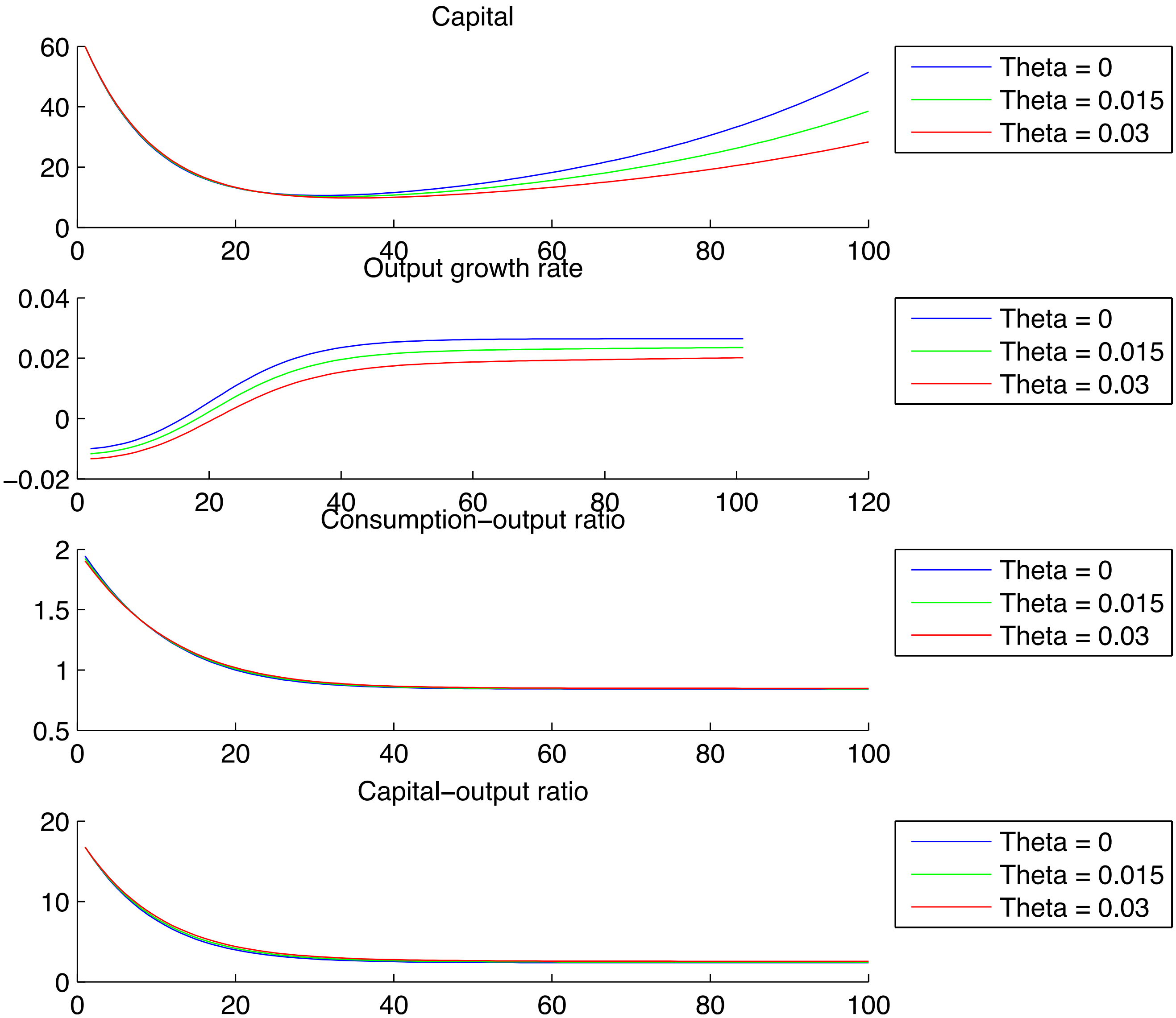
3.2.1 An example with $\gamma > 1$



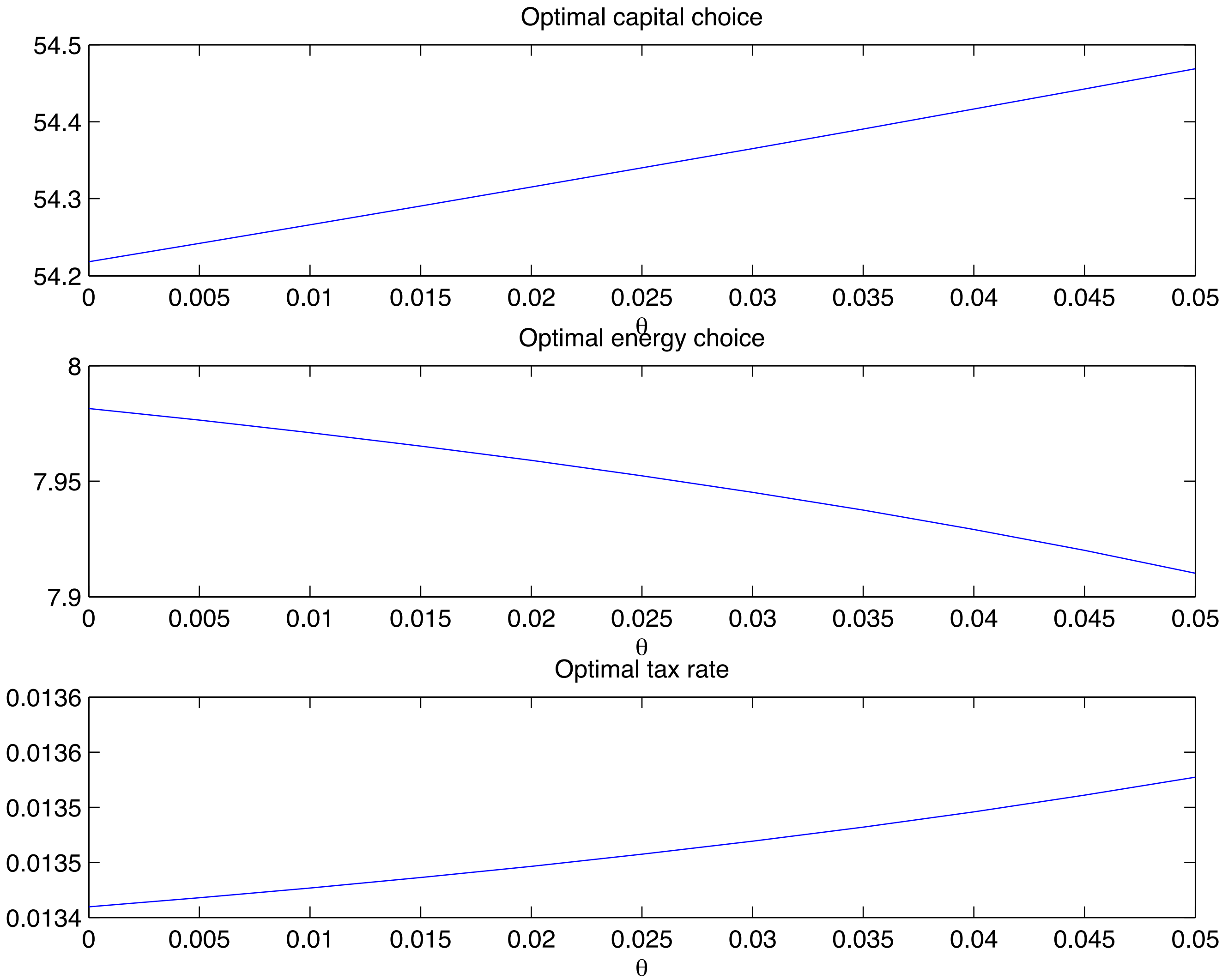
This figure plots the distortions to temperature, log productivity, and log damages that a robust agent worries about when $\gamma = 1.2$, for three different values of θ. When $\theta = 0$ the distortions are zero.



This figure plots the evolution of reserves, energy usage, the log of the optimal energy tax, and temperature when $\gamma = 1.2$, for three different values of θ.



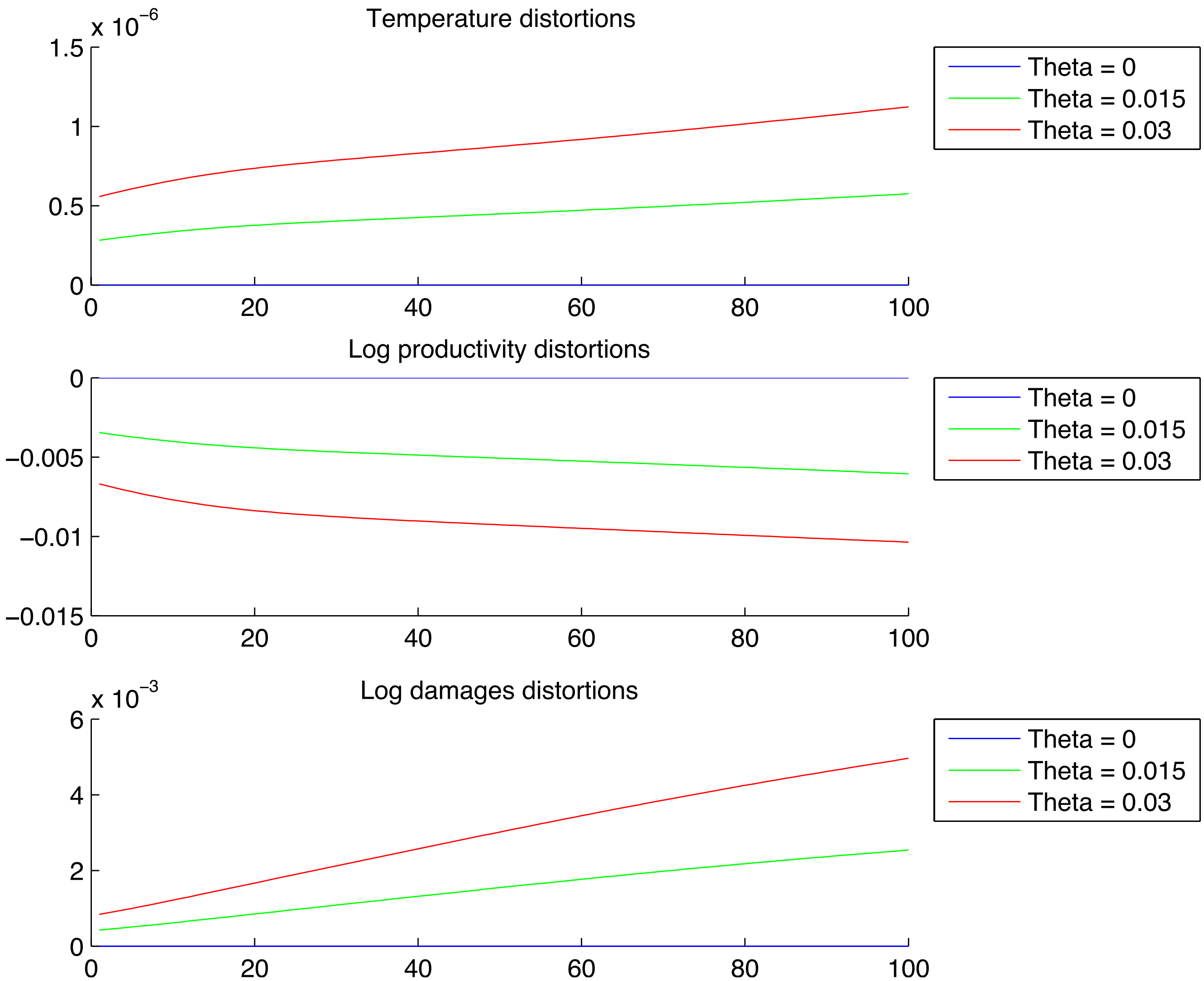
This figure plots the evolution of capital, the output growth rate, the consumption-output ratio, and the capital-output ratio when $\gamma = 1.2$, for three different values of θ.



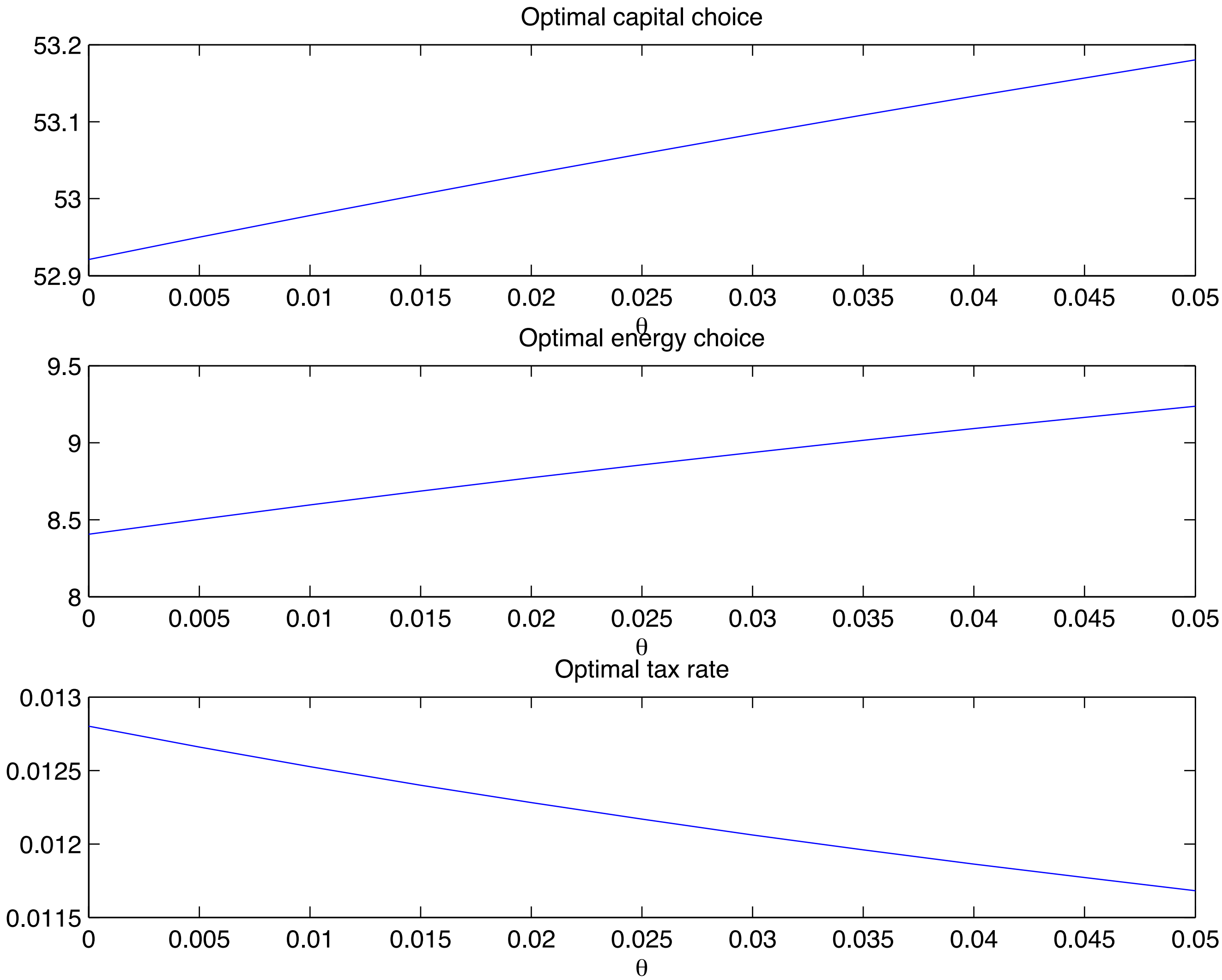
This figure plots the optimal capital decision rule, the optimal energy decision rule and the optimal tax rate as a function of θ, when $\gamma = 1.2$. The top graph is the optimal choice of next period's capital (K_1) chosen at date zero. The middle graph is the optimal choice of energy usage (E_0) at date zero. The bottom graph calculates the optimal tax rate on energy at time zero (which is $\nu Y_0/E_0$).

3.2.2 An example with $\gamma < 1$

In this Subsection we set $\gamma = 0.8$.



This figure plots the distortions to temperature, log productivity, and log damages that a robust agent worries about when $\gamma = 0.8$, for three different values of θ. When $\theta = 0$ the distortions are zero. Notice that distortions are steadily increasing over time. The rate of increase over time, rises when θ rises, for the temperature and log damages distortions.



This figure plots the optimal capital decision rule, the optimal energy decision rule and the optimal tax rate as a function of θ, when $\gamma = 0.8$. The top graph is the optimal choice of next period's capital (K_1) chosen at date zero. The middle graph is the optimal choice of energy usage (E_0) at date zero. The bottom graph calculates the optimal tax rate on energy at time zero (which is $\nu Y_0/E_0$).

3.3 Stochastic solutions

In this section, following Anderson, Hansen and Sargent (2012), we compute an approximation to the solution of a stochastic robust problem by expanding around the deterministic planner's problem. Let $V_0^{\epsilon,\theta}\left(K, R, \hat{T}, \log A, \log D\right)$ be the value function for the stochastic robust problem at time zero (in the finite horizon, discrete problem) when $\gamma = 1.2$. Here we parameterize the value function in terms of ϵ and θ in addition to the state variables. We compute the expansion for the finite horizon problem with $S = 600$.

We compute the following approximation for the value function:

$$V_0^{\epsilon,\theta}(60, 1000, 0, \log 1, \log 0.01) \approx -73.9134 - 23.3109\,\epsilon\theta - 0.2930\,\epsilon$$

where

- -73.9134 is the value of the agents problem when $\epsilon = 0$ and $\theta = 0$,
- -23.3109 is a robustness correction, and
- -0.2930 is a noise correction.

This is a first order (in ϵ) expansion around the deterministic planner's problem.

Let $K_{0,1}^{\epsilon,\theta}\left(K, R, \hat{T}, \log A, \log D\right)$ and $R_{0,1}^{\epsilon,\theta}\left(K, R, \hat{T}, \log A, \log D\right)$ be the optimal choices of time one capital and reserves (these decisions are made at time zero). We compute an approximation for the optimal decision rules as:

$$K_{0,1}^{\epsilon,\theta}(60, 1000, 0, \log 1, \log 0.01) \approx 54.218 + 4.7514\,\epsilon\theta + 0.0660\epsilon$$

$$R_{0,1}^{\epsilon,\theta}(60, 1000, 0, \log 1, \log 0.01) \approx 992.02 + 0.9804\epsilon\theta + 2.4202\epsilon$$

where

- 54.218 and 992.02 are the optimal choices when $\epsilon = 0$ and $\theta = 0$,
- $+4.7514$ and $+0.9804$ are robustness corrections, and

- $+0.0660$ and $+2.4202$ are noise corrections.

The approximate distortions induced by robustness to $\hat{T}_1, \log A_1$, and $\log D_1$ are:

$$
\begin{aligned}
&\text{Temperature:} && -\sqrt{\epsilon}\sigma_T G_{c0}^{\epsilon,\theta} && \approx \theta\epsilon\sigma_T^2 \frac{dV_1^{0,0}}{d\hat{T}_1} = 0.000011\,\theta\epsilon \\
&\text{Log productivity:} && -\sqrt{\epsilon}\sigma_a G_{a0}^{\epsilon,\theta} && \approx -\theta\epsilon\sigma_a^2 \frac{dV_1^{0,0}}{d\log A_1} = -0.13267\,\theta\epsilon \\
&\text{Log damages:} && -\sqrt{\epsilon}\sigma_d G_{d0}^{\epsilon,\theta} && \approx -\theta\epsilon\sigma_d^2 \frac{dV_1^{0,0}}{d\log D_1} = 0.0164\,\theta\epsilon
\end{aligned}
$$

where $V_1^{0,0}$ is deterministic planner's problem value function at time one when $\epsilon = 0$ and $\theta = 0$.

The expansions can be decomposed to characterize the effect of each of the shocks on the value of the planner's problem, optimal decisions, and distortions. Because of the linear nature of the expansions, when the expansions for the three shocks are added together, we recover the original expansion with all three shocks. Thus we can write:

$$
\begin{aligned}
V_0^{\epsilon,\theta,\sigma_T,\sigma_a,\sigma_d}(60, 1000, 0, \log 1, \log 0.01) &\approx -73.9134 \\
&- \epsilon\theta \begin{bmatrix} 2307.14 & 0.162 & 5.985 \end{bmatrix} \begin{bmatrix} \sigma_a^2 \\ \sigma_T^2 \\ \sigma_d^2 \end{bmatrix} + \epsilon \begin{bmatrix} -9.9200 & -0.000188 & -4.847 \end{bmatrix} \begin{bmatrix} \sigma_a^2 \\ \sigma_T^2 \\ \sigma_d^2 \end{bmatrix}
\end{aligned}
$$

where the expansion is now parameterized in terms of σ_a, σ_T and σ_d as well as θ and ϵ. This is the same first order (in ϵ) expansion around the deterministic planner's problem presented earlier – its just parameterized differently.

A decomposed approximation for the optimal capital decision rule is:

$$K_{0,1}^{\epsilon,\theta,\sigma_T,\sigma_a.\sigma_d}(60, 1000, 0, \log 1, \log 0.01) \approx 54.218 + \epsilon\theta \begin{bmatrix} 474.07 & 0.03645 & 0.268 \end{bmatrix} \begin{bmatrix} \sigma_a^2 \\ \sigma_T^2 \\ \sigma_d^2 \end{bmatrix} + \epsilon \begin{bmatrix} 6.6800 & 0.00050 & -0.020 \end{bmatrix} \begin{bmatrix} \sigma_a^2 \\ \sigma_T^2 \\ \sigma_d^2 \end{bmatrix}.$$

A decomposed approximation for the optimal reserves decision rule is:

$$R_{0,1}^{\epsilon,\theta,\sigma_T,\sigma_a.\sigma_d}(60, 1000, 0, \log 1, \log 0.01) \approx 992.02 + \epsilon\theta \begin{bmatrix} -258.98 & -0.02537 & 89.255 \end{bmatrix} \begin{bmatrix} \sigma_a^2 \\ \sigma_T^2 \\ \sigma_d^2 \end{bmatrix} + \epsilon \begin{bmatrix} -13.580 & -0.001119 & 63.90 \end{bmatrix} \begin{bmatrix} \sigma_a^2 \\ \sigma_T^2 \\ \sigma_d^2 \end{bmatrix}.$$

Decomposed approximate distortions induced by robustness to $\hat{T}_1, \log A_1$, and $\log D_1$ are:

$$\begin{aligned}
&\text{Temperature:} && -\sqrt{\epsilon}\sigma_T G_{c0}^{\epsilon,\theta,\sigma_T,\sigma_a} && \approx \theta\epsilon\sigma_T^2 \frac{dV_1^{0,0,0,0,0}}{d\hat{T}_1} = 0.11\,\theta\epsilon\sigma_T^2 \\
&\text{Log productivity:} && -\sqrt{\epsilon}\sigma_a G_{a0}^{\epsilon,\theta,\sigma_T,\sigma_a} && \approx -\theta\epsilon\sigma_a^2 \frac{dV_1^{0,0,0,0,0}}{d\log A_1} = -13.267\,\theta\epsilon\sigma_a^2 \\
&\text{Log damages:} && -\sqrt{\epsilon}\sigma_d G_{d0}^{\epsilon,\theta,\sigma_d,\sigma_d} && \approx -\theta\epsilon\sigma_d^2 \frac{dV_1^{0,0,0,0,0}}{d\log D_1} = 0.409\,\theta\epsilon\sigma_d^2.
\end{aligned}$$

The stochastic results provide some interesting links between robustness and optimal choices. For the parameter values used in this subsection, both noise (without robustness) and noise in conjunction with robustness lower lifetime utility. The amount of noise in productivity and damages (as measured by σ_a^2 and σ_d^2) has much larger effects on welfare and decisions than the amount of noise in temperature (as measured by σ_T^2).

We also see whether or not the reserves decision rule is increasing with θ (which implies energy usage is decreasing with θ) depends crucially on the magnitudes of σ_T, σ_a, and σ_d as well as γ. For the parameter values discussed in this section, with $\gamma = 1.2$, energy usage falls as σ_d increases but rises as σ_T and σ_a increases.

In this section, we considered robust preferences with power utility functions. Unless $\gamma = 1$, these preferences are non-homothetic. It would be interesting to compare the preferences in this section with Maenhout (2004)'s homothetic modification of robust power preferences.

3.4 Interpreting the results

We have shown that the effects of robustness can be large. We provided an example where optimal energy usage increases by 10% when $\theta = 0.05$. We also provided expansions which show the effects of robustness can (at least approximately) be very large when θ is large. By varying parameters, robustness can lead to more or less energy usage; and more or less precautionary savings. In this section, we use the analytical results of Section 2, to help understand the computational results. We make seven points about the relationship of the results.

First, the consumption-output ratio and the capital-output ratio approximately converge to constants as t increases in Section 3. Equation [2.3] predicts that these two ratios are constant for all t. Of course the Section 2 model is much simpler than the Section 3 model. E.g. the Section 2 model assumes full depreciation of capital in one period in contrast to the Section 3 model. Nevertheless it is instructive that these two ratios approximately converge to constants in the more complicated Section 3 model consistent with the predictions of the Section 2 model.

Second, the Section 2 model predicts that robustness does not change the consumption-output ratio and the capital output ratio. We see the same result in Section 3 where the robustness effects on these two ratios are very small.

Third, the Section 2 model, equation [2.9] predicts that the log difference between output and carbon tax at date t is a constant independent of the robustness parameter, θ (when it is constant in t). This appears to be approximately true in the plots in Section 3 as can be seen by taking the log of the capital plot and comparing it to the log of the energy tax plot. (In order for it to be exactly true, energy usage would have to be constant over time.)

Fourth, the paths of reserves, energy usage, and temperature appear roughly independent of robustness in the Section 3 plots. Equation [2.8] shows that energy usage is approximately independent of robustness because the only channel where robustness may have an effect is through the shadow price of the reserve.

Fifth, we observed in Section 2 that the energy tax may well decrease in robustness (which is observed in the Section 3 energy tax plot). This effect occurs in [2.9] because output falls as robustness increases under the worst case model. However, [2.9] predicts that robustness has no effect on the ratio of the energy tax to output. This ratio is predicted to be approximately constant (and constant when Δ_t is constant in t) by [2.9]. This last point is the same as the third point above.

Sixth, the output growth rate approximately converges to a constant which falls as robustness increases in the Section 3 plot. The growth rate is constant and falls as robustness increases in [2.10] (except for the stochastic shocks).

Seventh, the trend term $\lambda b E_t$ in the temperature dynamics [2.12] is approximately independent of robustness which only enters the fluctuation-about-trend term, $\hat{T}_{t+1} - \hat{T}_t$ through the distortion effect caused by robustness. The relatively small effect of robustness on the trend over time in temperature is seen in the Section 3 plot of temperature over time.

It is clear by now that the simple analytical model of Section 2 is very useful in helping one understand the forces behind the findings in Section 3 even though the Section 3 model is more complicated. There are more points that could be made but we stop here in order to save space.

4 Summary, Conclusions, and Future Research

This paper has presented an initial version of an approach to building simple dynamic integrated assessment models in which the representation of climate dynamics is based on the cumulative carbon response (CCR) metric developed by Matthews et al. (2009, 2012). As these researchers argue, this reduced form framework is both more relevant, and more directly applicable, to the core policy problems of GHG emissions abatement than the standard representation based on the equilibrium climate sensitivity parameter. Among the advantages of the CCR is that it appears to have a lower level of uncertainty than the sensitivity parameter, the distribution of which continues to be a matter of debate despite many years of work by numerous researchers (Roe and Baker (2007)).

In Section 1 of this paper we presented the logic and evidence supporting the use of the CCR in this context, and briefly reviewed the integrated assessment literature focusing on uncertainty analysis to which our work makes a contribution. Section 2 presented a simple analytical model for which closed form solutions can be developed and the impacts of changes in the CCR, robustness parameters, and other key parameters can be readily derived and compared. These analytical results help us understand the basic forces behind the computational results in Section 3, for a more complicated model where closed form solutions are not available. We generate results for two different values of the risk aversion parameter, one which is less than one, the other which is greater than one. The value is studied in Section 2. We also display plots for several different values of the robustness parameter.

We wish to emphasize that our paper reports an initial and incomplete analysis that nonetheless appears to be the first to apply the cumulative carbon response approach as the basis of economic and policy analysis in dynamic coupled climate/economic modeling. We therefore believe it to be a useful contribution to the large received literature on integrated assessment modeling, and plan to develop this approach much more extensively in the near future. For example, we believe that a robustness analysis that includes spatial aspects of climate dynamics (Brock, Engstrom and Xepapadeas (forthcoming)) that is much more disaggregated than our current work is needed since many concerns about damages and other effects of changing climate are regional.

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About RDCEP

The Center brings together experts in economics, physical sciences, energy technologies, law, computational mathematics, statistics, and computer science to undertake a series of tightly connected research programs aimed at improving the computational models needed to evaluate climate and energy policies, and to make robust decisions based on outcomes.

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