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A Multi-Modal Analysis of Climate-Economics

Emmanuel Souganidis

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Emmanuel Souganidis†

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Abstract

We investigate the qualitative properties of the climate-economic growth model introduced by Brock et al., (2012). We assume that the mean annual distribution of solar radiation energy and the fraction of incoming radiation flux absorbed by the surface have a specific form, and perform a rigorous mathematical analysis when the time scale for temperature is taken to be faster than that of carbon. We analyze the impact of moving welfare weights away from Negishi weights by introducing a simple welfare weights function. We perform a qualitative analysis on the output elasticities of carbon and capital, and examine the way thermal diffusion affects local economic variables and taxes at the equilibrium. We then evaluate the robustness of the two- and four-mode forms in the context of the temperature model.

1 Introduction

We investigate the qualitative properties of the climate-economic problem introduced by Brock et al., (2012). In their paper, Brock et al., (2012) couple a spatial climate model with a model of economic growth. They follow North (1975a) and write the mean annual distribution of solar radiation energy and the co-albedo function, i.e., the fraction of incoming radiation flux absorbed by the surface, in a two-mode form. They then assume that the solution of the climate model is of the same (two-mode) form and continue with their analysis.

In this paper, we analyze the validity and significance of using different functional forms for the mean annual distribution of solar radiation energy and the co-albedo function by comparing their impact on the economic model. Using input functions of two- and four-mode forms, we obtain, under the assumption that temperature grows faster than carbon, the exact solutions of the climate problem in a mathematically rigorous way. In doing so we observe that the exact solution in the two-mode case is given by four modes[1], and the solution in the four-mode case is given by eight modes.

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†Associate Economist, Federal Reserve Bank of Chicago. The opinions expressed herein are my own and do not necessarily represent those of the Federal Reserve Bank of Chicago or the Federal Reserve System.

[1]The presence of the fourth mode in the solution of the temperature problem with two-mode form input functions was previously neglected in Brock et al., (2012).

We use these solutions with the economic problem in Brock et al., (2012) to evaluate the change in optimal taxation policy as one moves away from Negishi welfare weights (Stanton (2009)). We determine the dependency of temperature and damages on the rate of diffusion and analyze the qualitative impact of the output elasticities of capital and carbon on the socially optimal quantities at the steady state. We evaluate the robustness of the two- and four-mode forms and find that the impact of thermal diffusion on the spatial distributions of temperature and damages behaves differently for the two cases. We observe that, as the rate of diffusion increases, the area in the world for which damages are reduced increases for the two-mode case and decreases for the four-mode case. This differs from the observation in Brock et al., (2012) that the size of the area does not change.

In their paper, Brock et al., (2012) determine an optimal mitigation policy to correct for the climate externality in both spatially uniform and spatially differentiated settings. When international transfers are allowed, their results indicate that a spatially uniform carbon tax can emerge with the implementation of Negishi welfare weights (Stanton (2009)). Alternatively, Chichilnisky and Heal (1994) note that in the absence of international transfers, a spatially uniform optimal mitigation policy is not possible. Here we introduce a simple welfare weights function and analyze the impact of moving away from the Negishi weights. Following Brock et al., (2012), we also investigate the impact of thermal diffusion on the spatial distribution of fossil fuel taxes given our welfare weight function in the polar case of closed economies. Our results confirm that an increase in the rate of thermal diffusion will decrease the optimal carbon tax rate for countries close to the equator and increase it for those further away.

We then evaluate qualitatively the impact of carbon and capital output elasticities on the steady state distributions of temperature, damages, capital, and social price of carbon. These output elasticities can be used to represent various climate-economic policies. For example, an increase in the output elasticity of carbon would indicate a movement away from capital-heavy production and more towards carbon-heavy production. Thus, when viewed in the context of industrial policy, our results express the relationship between environmental regulation and the social planner problem.

Our final objective is to evaluate the robustness of the two- and four-mode forms of the mean annual distribution of solar radiation energy and the co-albedo function in the context of dynamic paths and solutions to the social planner's optimal control problem presented in Brock et al., (2012). As North et al., (1981) have shown in discrete settings, the static temperature distribution at any fixed point in time is well approximated by a two-mode expansion. Our results confirm that the two-mode form adequately models the steady state distributions when compared to the four-mode form through a distance metric. We therefore believe this to be one of the main contributions of our paper relative to the analysis previously completed for the discrete case.

The paper is organized as follows: In Section 2 we review the basic one-dimensional energy balance climate model with human inputs introduced by Brock et al., (2012). In Section 3 we provide a rigorous mathematical derivation for the temperature distribution using two- and four-mode forms for the mean annual distribution of solar radiation energy and the co-albedo function, and determine numerically the exact latitude dependent temperature functions. In Section 4, building on Brock et al., (2012) and Kopp et al., (2012), we introduce an exponential damage function to analyze the impact of climate change. We also provide solutions to the social planner problem and the problem of competitive equilibrium in the world market. In Section 5 we discuss the optimal mitigation policy for carbon in both spatially uniform and spatially differentiated cases, and introduce a simple function for welfare weights. In Section 6 we show how heat transport affects the steady state distributions of local economic variables. In Section 7 we analyze qualitatively the effect of carbon and capital elasticities on the steady state solutions. In Section 8 we investigate the robustness of the two- and four-mode modes by comparing the exact solutions to their respective temperature models. In Appendix A and Appendix B we derive the exact solutions to the temperature problem for the two- and four-mode cases, respectively. In Appendix C we derive the steady state distributions for the socially optimal quantities.

2 Environmental Model Details

Here we recall the one-dimensional energy balance climate model (EBCM) with human inputs introduced by Brock et al., (2012). There are of course other climate models (e.g. Nordhaus (2007a,b), (2010), (2011)) which provide a spatial distribution of damages, but these are based on more complex and computationally costly models, such as pattern scaling (Lopez et al., (2012)) or emulation theory (e.g. Challenor et al., (2006)). In this note we concentrate on the simplest coupled climate-economic model. The explicit one-dimensional spatial property allows the coupled climate-economic model to evolve in both time and space. In the presentation, we follow Brock et al., (2012) and use their notation.

Let x denote the *sine* of the latitude; then $x = -1$, $x = 0$, and $x = 1$ mark the South Pole, Equator, and North Pole, respectively. For simplicity, we refer to x as latitude. Let $T(x,t)$ denote the sea level temperature measured in °C at latitude x and time t. The basic energy balance equation with human input added can be written as (Wu and North (2007))

$$C_c \frac{\partial T(x,t)}{\partial t} = QS(x)\alpha(x, x_s(t)) - (A + BT(x,t) - h(t)) + D\frac{\partial}{\partial x}\left((1-x^2)\frac{\partial T(x,t)}{\partial x}\right), \quad (1)$$

where C_c denotes the effective heat capacity per unit area of earth atmospheric system (large over water, small over land), $2Q$ is the solar constant, $S(x)$ is the mean annual distribution of solar radiation energy, $\alpha(x, x_s(t))$ is the absorption coefficient or co-albedo function, which is one minus the albedo of the earth-atmosphere system, with $x_s(t)$ being the latitude of the ice line at time t, $A + BT$ is the rate of outgoing infrared radiation to space with the empirical coefficients A and B derived from satellite measurements[2], $h(t)$ is the human input at time t which reduces the amount of outgoing radiation (green-house effect), and D is a heat transport coefficient measured in $W/(m^2)(°\text{C})$.

Human input is represented by the amount of accumulated carbon dioxide in the atmosphere that reduces outgoing radiation. It is defined by

$$h(t) = \xi \ln\left(1 + \frac{M(t)}{M_0}\right), \quad (2)$$

[2]The coefficients A and B take into account average cloudiness conditions, the effects of infrared absorbing gases, and the variability of water vapor (North et al., (1975)).

Table 1: Parameter Values

Parameter	Value	Parameter	Value
α_0	0.681	Q	688 Wm^{-2}
α_2	-0.202	M_0	596 GtC
α_4	-0.037	M_{2011}	831 GtC
S_0	0.5	B	1.24 $W(m^{-1})(^\circ C^{-1})$
S_2	-0.2385	D	0.3 $W(m^{-1})(^\circ C^{-1})$
S_4	-0.045	ξ	5.35 $^\circ Wm^{-2}$
A	221.6 Wm^{-2}	g	1.178% IPCC A1F1

The parameter values for the dimensionless $\alpha_0, \alpha_2, \alpha_4, S_0, S_2, S_4$ have been obtained by North et al., (1981). The parameter values for A, B, D have been obtained by calibration so as to reproduce current global temperature (Brock et al., (2012)). $g = 1.178\%$ is the average annual growth of carbon emissions corresponding to the IPCC scenario $A1F1$ (http://www.ipcc-data.org/sres/ddc_sres_emissions.html).

where M_0 is the pre-industrial concentration of atmospheric carbon dioxide (CO_2), $M(t)$ is the concentration of CO_2 at time t, and $\xi = 5.35^\circ Wm^{-2}$ is a temperature-forcing parameter ($^\circ$C per W per m^2). The function $M(t)$, which should be interpreted as the stock of man-made CO_2 in the atmosphere, evolves according to

$$\dot{M}(t) = \int_{-1}^{1} \beta q(x,t)dx - mM(t) \quad \text{and} \quad M(0) = M_0, \tag{3}$$

where $\beta q(x,t)$ are the emissions generated at time t and are assumed to be proportional to the amount of fossil fuels used at latitude x at time t.

We assume that the total stock of fossil fuel available is fixed, i.e.,

$$\int_{-1}^{1} q(x,t)dx = q(t) \quad \text{and} \quad \int_{0}^{\infty} q(t) = R_0, \tag{4}$$

where $q(t)$ is total fossil fuels used across all latitudes at time t, and R_0 is the total available amount of fossil fuels on the planet.

The greenhouse effect is thus incorporated in this model. The use of fossil fuels generates emissions which increase the stock of atmospheric CO_2. This carbon dioxide increases the temperature by blocking the outgoing radiation.

At equilibrium, the incoming absorbed radiant heat at a given latitude in (1) is not matched by the net outgoing radiation. The difference is made by the meridional divergence of heat flux, which is modeled by the term $D\frac{\partial}{\partial x}\left((1-x^2)\frac{\partial T(x,t)}{\partial x}\right)$ (North et al., (1981)). This term explicitly introduces into the climate model the spatial dimension stemming from heat diffusion.

The ice line is determined dynamically by the condition (Budyko (1969) and North (1975a,b))

$$\begin{cases} T > T_s & \text{no ice line present at latitude } x, \\ T < T_s & \text{ice present at latitude } x, \end{cases} \tag{5}$$

where T_s is empirically determined.

In general, the co-albedo function specified by North (1975a) is

$$\alpha(x, x_s) = \begin{cases} \alpha_0 = 0.38 & \text{if} \quad |x| > x_s, \\ \alpha_1 = 0.68 & \text{if} \quad |x| < x_s; \end{cases} \tag{6}$$

note that since the albedo jumps discontinuously below the ice line, so does the absorption.

This general form of α presents serious mathematical difficulties. As a result, it was proposed by North et al., (1981) and was assumed by Brock et al., (2012) that α and S are given by

$$\alpha(x) = \alpha_0 + \alpha_2 P_2(x) \quad \text{and} \quad S(x) = S_0 + S_2 P_2(x), \tag{7}$$

where P are the canonical Legendre polynomials recalled in Appendix A and B. North et al., (1981) claim that this expression is sufficient. Here we rigorously test this assumption and compare it to the four-mode case

$$\alpha(x) = \alpha_0 + \alpha_2 P_2(x) + \alpha_4 P_4(x) \quad \text{and} \quad S(x) = S_0 + S_2 P_2(x) + S_4 P_4(x). \tag{8}$$

In general, α and S can be written as

$$\alpha(x) = \sum_{k\geq 0} \alpha_{2k} P_{2k}(x) \quad \text{and} \quad S(x) = \sum_{k\geq 0} S_{2k} P_{2k}(x), \tag{9}$$

where $P_{2k}(x)$ are even-numbered Legendre polynomials, but we do not pursue this here.

3 Mathematical Derivations

In view of the form of (1) and the general expression for α and S, to solve equation (1) it is enough to assume that the solution is of the form

$$\hat{T}(x, t) = \sum_{k\geq 0} T_{2k}(t) P_{2k}(x), \tag{10}$$

where $T_{2k}(t)$ are solutions to appropriately derived ODEs. Note that we consider only even-numbered Legendre polynomials to preserve the symmetry assumption for latitude x.

To simplify our analysis, we further assume that T evolves in time scale faster than M and, hence, relaxes faster to the steady state. In mathematical terms, this means that $\frac{\partial T}{\partial t} = 0$, i.e., $\frac{\partial T_{2k}}{\partial t} = 0$ for all k.

From this assumption, we find that T solves

$$A + BT(x, t) = h(t) + QS(x)\alpha(x) + D\frac{\partial}{\partial x}\left((1 - x^2)\frac{\partial T(x, t)}{\partial x}\right). \tag{11}$$

The goal is then to identify the coefficients T_{2k} and, thus, T for the two- and four-mode cases.

It turns out that in the two-mode case $\hat{T}$ is expressed in terms of P_0, P_2, and P_4, while in the four-mode case $\hat{T}$ is expressed in terms of P_0, P_2, P_4, P_6 and P_8. This observation is one of the key contributions of the paper, since in previous work (in the two-mode case) $\hat{T}$ was taken to have only two modes.

3.1 Two-Mode

We assume that α and S are given by (7). Then, as shown in Appendix A, the exact solution to the temperature PDE (11) is

$$\hat{T}(x,t;D) = T_0(t) + T_2(D)P_2(x) + T_4(D)P_4(x), \tag{12}$$

where

$$T_0(t) = \frac{1}{B}\left(-A + Q\left(\frac{1}{5}\alpha_2 S_2 + \alpha_0 S_0\right) + h(t)\right), \tag{13}$$

$$T_2(D) = \frac{Q}{(B+6D)}\left(\frac{2}{7}\alpha_2 S_2 + \alpha_0 S_2 + \alpha_2 S_0\right), \tag{14}$$

and

$$T_4(D) = \frac{Q}{(B+20D)}\left(\frac{18}{35}\alpha_2 S_2\right). \tag{15}$$

From (13)-(15) and (2), we see that T_0 depends on the concentration M but not on the thermal transport coefficient D, and conversely, that T_2 and T_4 depend on D but not on M. For future reference, it is convenient to write the temperature field as

$$\hat{T}_{[2]}(x,t) = Z_{[2],0} + Z_{[2],1}\,\ln\left(1 + \frac{M(t)}{M_0}\right) + \frac{Z_{[2],2}}{B+6D}P_2(x) + \frac{Z_{[2],4}}{B+20D}P_4(x), \tag{16}$$

where $(_{[2]})$ refers to the two-mode form. From the parameter values in Table 1, we find that

$$Z_{[2],0}, Z_{[2],1}, Z_{[2],4} > 0 \quad \text{and} \quad Z_{[2],2} < 0. \tag{17}$$

The temperature function $\hat{T}_{[2]}$ is shown in Figure 1 with $t = 0$ corresponding to year 2011.

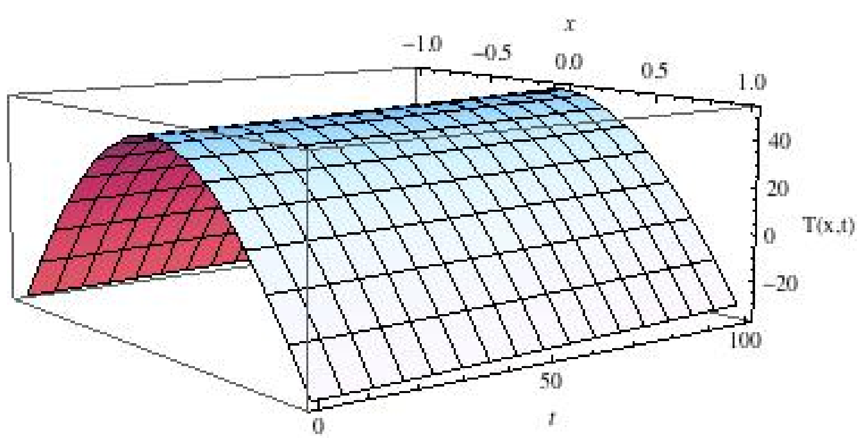

Figure 1: The temperature function

The predicted temperature increases over a horizon of 100 years.

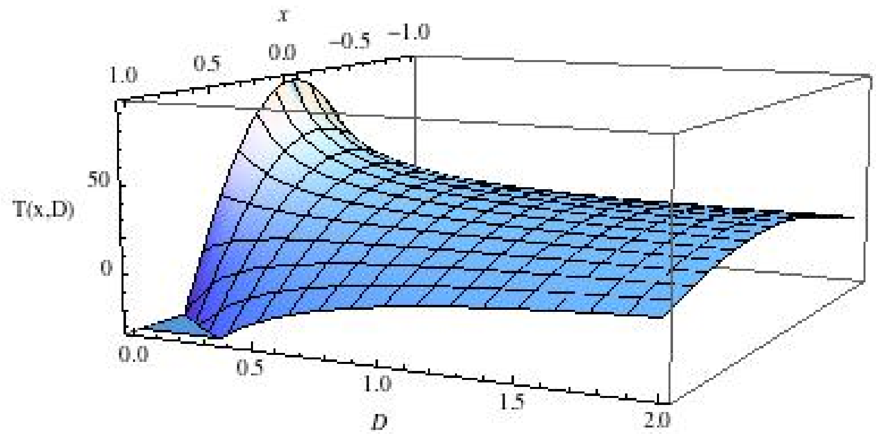

Figure 2: Temperature and thermal transport

Figure 2 depicts temperature as a function of latitude and the thermal transport coefficient D. When $D \to \infty$, the temperature function becomes spatially homogenous or "flat" across latitudes, as the increase in transport speeds results in homogeneity.

Thus, in the two-mode case, an increase in D will warm x near the North and South Poles more and cool x near the equator more for a given increase in $M(t)$, as in Figure 2 (or Figure 2 of Brock et al., (2012) and Figure 4 of North et al., (1981)).

3.2 Four-Mode

We use the four-mode expression for α and S in (8). It is shown in Appendix B that the exact solution to the temperature PDE (11) is

$$\hat{T}(x,t;D) = T_0(t) + T_2(D)P_2(x) + T_4(D)P_4(x) + T_6(D)P_6(x) + T_8(D)P_8(x), \tag{18}$$

where

$$T_0(t) = \frac{1}{B}\left(-A + Q\left(\frac{1}{5}\alpha_2 S_2 - \frac{1193}{288}\alpha_4 S_4 + \alpha_0 S_0\right) + h(t)\right), \tag{19}$$

$$T_2(D) = \frac{Q}{B+6D}\left(\frac{2}{7}(\alpha_2 S_4 + \alpha_4 S_2) - \frac{42475}{2772}\alpha_4 S_4 + \frac{2}{7}\alpha_2 S_2 + \alpha_0 S_2 + \alpha_2 S_0\right), \tag{20}$$

$$T_4(D) = \frac{Q}{B+20D}\left(\frac{20}{77}(\alpha_2 S_4 + \alpha_4 S_2) - \frac{25401}{2002}\alpha_4 S_4 + \frac{18}{35}\alpha_2 S_2 + \alpha_0 S_4 + \alpha_4 S_0\right), \tag{21}$$

$$T_6(D) = \frac{Q}{B+42D}\left(\frac{5}{11}(\alpha_2 S_4 + \alpha_4 S_2) - \frac{470}{99}\alpha_4 S_4\right), \tag{22}$$

and

$$T_8(D) = \frac{Q}{B+72D}\left(\frac{490}{1287}\alpha_4 S_4\right). \tag{23}$$

From (19)-(23) and (2), we see that T_0 depends on the concentration M but not on the thermal transport coefficient D, and, conversely, that T_2, T_4, T_6, and T_8 depend on D but not on M. Hence, it is convenient to write the temperature field as

$$\hat{T}_{[4]}(x,t) = Z_{[4],0} + Z_{[4],1} \ln\left(1 + \frac{M(t)}{M_0}\right) + \frac{Z_{[4],2}}{B+6D} P_2(x) + \frac{Z_{[4],4}}{B+20D} P_4(x) + \frac{Z_{[4],6}}{B+42D} P_6(x) + \frac{Z_{[4],8}}{B+72D} P_8(x), \tag{24}$$

where $({}_{[4]})$ refers to the four-mode form. From the parameter values in Table 1, we find that

$$Z_{[4],0}, Z_{[4],1}, Z_{[4],8} > 0 \quad \text{and} \quad Z_{[4],2}, Z_{[4],4}, Z_{[4],6} < 0. \tag{25}$$

The temperature function $\hat{T}_{[4]}$ is shown in Figure 3 with $t = 0$ corresponding to year 2011.

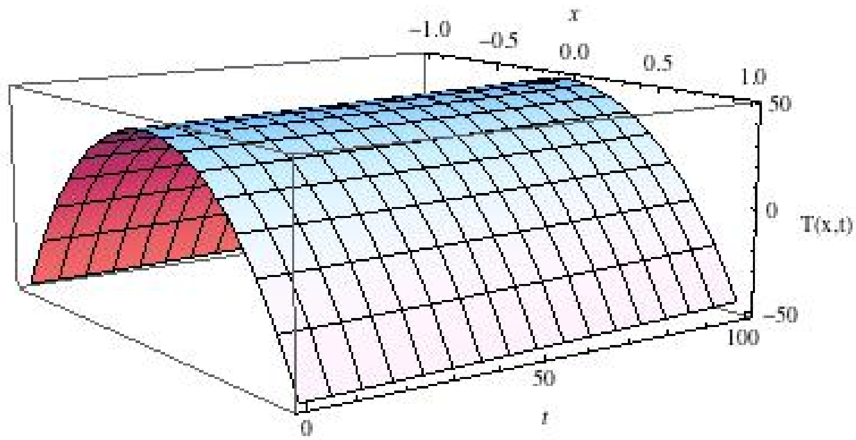

Figure 3: The temperature function

The predicted temperature increases over a horizon of 100 years.

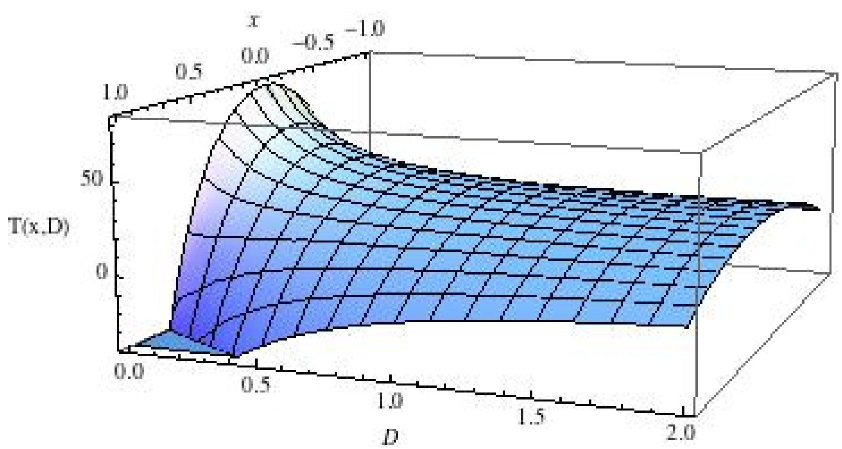

Figure 4: Temperature and thermal transport

Figure 4 depicts temperature as a function of latitude and the thermal transport coefficient D. When $D \to \infty$, the temperature function becomes spatially homogenous or "flat" across latitudes, as the increase in transport speeds results in homogeneity.

In the four-mode case, an increase in D will warm x near the North and South Poles more and cool x near the equator more for a given increase in $M(t)$, as in Figure 4 (or Figure 2 of Brock et al., (2012) and Figure 4 of North et al., (1981)).

4 Economic Energy Balance Climate Model

For completeness, we describe next the coupled climate-economic model introduced by Brock et al., (2012).

4.1 Damages

Since we are interested in the implications of thermal transport across latitudes we define damages in terms of the temperature distribution. In this way we can trace the impact of thermal transport on damages and perform meaningful comparative statistics with respect to the thermal transport coefficient, D.

As in Brock et al., (2012), we consider, for $\gamma > 0$, the exponential damage function

$$\Omega(\hat{T}(x,t)) = \exp\left(-\gamma \hat{T}(x,t)\right). \tag{26}$$

In the context of this problem, Ω denotes the proportion of GDP available at latitude x and time t after damages due to climate change have been accounted for. The elasticity of marginal damages with respect to the temperature is $-\gamma$, so an increase in temperature will increase damages when adaption is fixed. In mathematical terms, this means that $\frac{\partial \Omega}{\partial T} < 0$.

4.2 Local Output

We assume that the output of our economy, $Y(x,t)$, at latitude x and time t is produced according to a standard Cobb-Douglas production function, F, with constant returns to scale and constant total factor productivity (TFP). For mathematical simplicity, we assume no population growth and constant labor.

It then follows that

$$\begin{aligned} Y(x,t) &= \mathbb{A}\Omega(T(x,t))F(K(x,t), L, q(x,t)) \\ &= \mathbb{A}L^{\alpha_L}\Omega(T(x,t))K(x,t)^{\alpha_K}q(x,t)^{\alpha_q} \\ &= \Psi(x, T(x,t))K(x,t)^{\alpha_K}q(x,t)^{\alpha_q}, \end{aligned} \tag{27}$$

where

$$\Psi(x, T(x,t)) = \mathbb{A}L^{\alpha_L}\Omega(T(x,t)). \tag{28}$$

Here $\mathbb{A}$ and L are the TFP and labor, respectively, $K(x,t)$ and $q(x,t)$ denote capital and fossil fuels respectively used at latitude x and time t, α_K, α_L, and α_q denote output elasticities of capital, labor, and fossil fuels, respectively, and Ω is as in (26). The assumption of constant returns to scale implies that

$$\alpha_K + \alpha_L + \alpha_q = 1. \tag{29}$$

A more rigorous investigation on the impact of labor should look to model the following distribution:

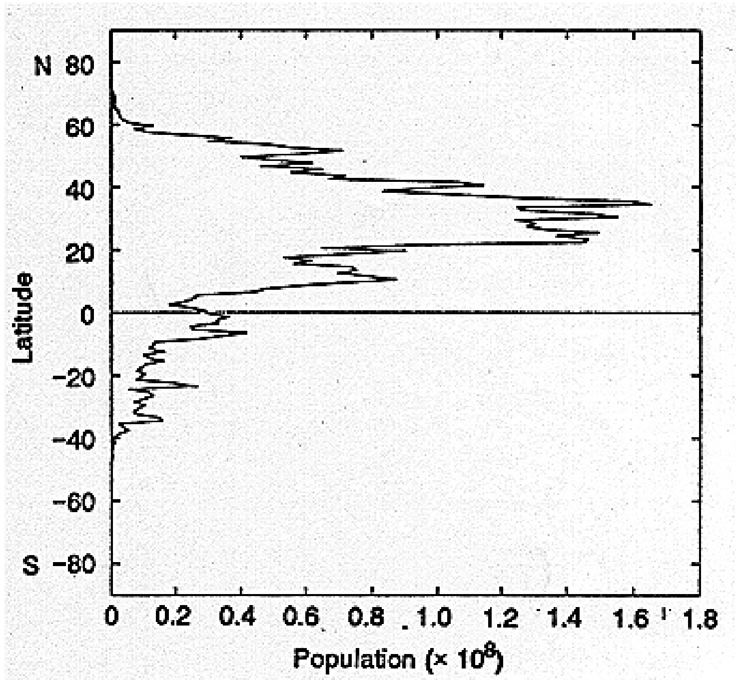

Figure 5: Latitudinal distribution of the world's population in 1990 from the Carbon Dioxide Information Analysis Center (CDIAC).

The population distribution is asymmetric about the equator and skewed towards the Southern Hemisphere. Data on labor participation rates[3] and population growth rates[4] from The World Bank can be used to scale the above distribution and model the evolution over time, respectively.

4.3 Potential World Output and Damages From Climate Change

In this economy we denote by $F_{total}(K(t), q(t), T; t)$ the maximum output that the whole world can produce. This takes into account the total world capital, $K(t)$, available and total world fossil fuel, $q(t)$, used, for a given distribution of temperature, T, across the globe. Note that for any function $\eta(x,t)$ (e.g. $\eta = C, K, q$), we abuse notation and, for the rest of the paper, write $\eta(t) = \int_X \eta(x,t)dx$, where $X = [-1, 1]$.

The overall resource constraint for the economy is

$$C(t) + \dot{K}(t) + \delta K(t) = F_{total}(K(t), q(t), T; t), \tag{30}$$

where $C(x,t)$ is the consumption at location x and time t, $C(t) = \int_X C(x,t)dx$ is the total consumption, and δ is the rate of depreciation, which is taken to be constant across latitudes.

[3]http://data.worldbank.org/indicator/SL.TLF.CACT.ZS
[4]http://data.worldbank.org/indicator/SP.POP.GROW

Our approach does not provide a breakdown of depreciation rates due to varying climate conditions, but this can be accommodated by modeling depreciation rates across latitudes. We leave this project for future research[5].

Using Ψ from (27), we define the specific (at location x and time t) and global damages, respectively,

$$J(x,t;D) = \frac{\Psi(x,T(x,t))^{1/\alpha_L}}{\left(\int_X \Psi(x',T(x',t)^{1/\alpha_L})dx'\right)^{\alpha_K+\alpha_q}}, \tag{31}$$

and

$$J(t;D) = \int_X J(x,t;D)dx. \tag{32}$$

The potential world GDP, $F_{total}(K(t),q(t),T;t)$, is computed through the following optimization problem

$$F_{total}(K(t),q(t),T;t) = \max \int_X \Psi(x,T(x,t))K(x,t)^{\alpha_K} q(x,t)^{\alpha_q} dx, \tag{33}$$

subject to the constraints

$$\int_X K(x,t)dx \leq K(t) \quad \text{and} \quad \int_X q(x,t)dx \leq q(t). \tag{34}$$

The Lagrangean associated with (33) is

$$\begin{aligned} \mathcal{L} = &\int_X \Psi(x,T(x,t))K(x,t)^{\alpha_K} q(x,t)^{\alpha_q} dx \\ &+ \mu_K(t)\left(K(t) - \int_X K(x,t)dx\right) + \mu_q(t)\left(q(t) - \int_X q(x,t)dx\right), \end{aligned} \tag{35}$$

with optimality conditions

$$\alpha_K \Psi(x,T(x,t))K(x,t)^{\alpha_K-1} q(x,t)^{\alpha_q} = \mu_K(t), \tag{36}$$

and

$$\alpha_q \Psi(x,T(x,t))K(x,t)^{\alpha_K} q(x,t)^{\alpha_q-1} = \mu_q(t). \tag{37}$$

In the context of potential world GDP, this means that the marginal products of capital and fossil fuels are equated across latitudes for all times t.

4.4 Global Welfare Maximization

We analyze the welfare maximization problem of a social planner in the context of the coupled climate-economic model in Brock et al., (2012).

[5]For example, to investigate the impact of climate on depreciation rates, one could take into account that in really cold, high latitudes, equipment is highly stressed, whereas in extremely warm, humid, and low latitudes, equipment rusts and malfunctions more frequently.

Allowing for per capita damages in utility due to climate change, $\Omega_C(T(x,t))$, turns the economic part of the social welfare model into the Ramsey-like form for the "aggregate dynastic consumer family." The optimization problem then becomes

$$\max \int_0^\infty \int_X v(x) L \left(U\left(\frac{C(x,t)}{L}\right) - \Omega_C(T(x,t)) \right) dx\, dt. \tag{38}$$

The planner maximizes (38) subject to the temperature climate constraint, the resource constraint in each location, (30), and the carbon constraint, $R_0(x) = \int_0^\infty q(x,t)dt$, where $v(x)$ are exogenously given non-negative welfare weights normalized such that $\int_X v(x)dx = 1$.

In Section 3 we established for the regime under investigation that

$$\hat{T}(x,t) = T_{[2],0}(t) + T_{[2],2}(D)P_2(x) + T_{[2],4}(D)P_4(x) \tag{39}$$

and

$$\hat{T}(x,t) = T_{[4],0}(t) + T_{[4],2}(D)P_2(x) + T_{[4],4}(D)P_4(x) + T_{[4],6}(D)P_6(x) + T_{[4],8}(D)P_8(x), \tag{40}$$

for the two- and four-mode forms of α and S respectively.

In view of (39) and (40), $T_{[2],0}$ and $T_{[4],0}$ are the only coefficients that have t as a parameter. Thus, the temporal impact of temperature on the social planner's problem is seen exclusively through $T_{[2],0}$ and $T_{[4],0}$ for the two- and four-mode cases, respectively.

In our context, the current value Hamiltonian for the social planner's problem is

$$\begin{aligned} H = & \int_X v(x) L \left(U\left(\frac{C(x,t)}{L}\right) - \Omega_C(\hat{T}(x,t)) \right) dx \\ & + \int_X \lambda_K(x,t) \left(\mathbb{A}\Omega(\hat{T}(x,t)) F(K(x,t), L, q(x,t)) - C(x,t) - \delta K(x,t) \right) dx \\ & + \lambda_M(t) \left(-mM(t) + \beta \int_X q(x,t)dx \right) - \mu_R(x,t) q(x,t) \\ & + \lambda_{T_0}(t) \left(-T_0 + Z_1 \ln\left(1 + \frac{M(t)}{M_0}\right) + Z_0 \right), \end{aligned} \tag{41}$$

where λ_K, λ_M, and λ_{T_0} are the Lagrange multipliers for capital, fossil fuels, and temperature respectively. The state and controls are, respectively, $\mathbf{v} = (K(t), R(t), M(t), T(t,x))$ and $\mathbf{u} = (C(t), C(x,t), q(t), q(x,t))$.

Since problem (38) is non-autonomous, we assume that the discount rate is sufficiently high and that the functions of the problem satisfy the growth conditions required to apply the Pontryagin maximum principle (Malysh, (2008) and Iyanaga, (1980)), which yields that the controls must satisfy

$$\begin{cases} \lambda_K(t) = \mu_C(t) = v(x) U'\left(\frac{C(x,t)}{L}\right), & (42) \\ \lambda_K(t) F'_{total,q} = \mu_R(t) - \mu_q(t), & (43) \\ \lambda_M(t)\beta = \mu_q(t). & (44) \end{cases}$$

Immediately one sees that, for equal welfare weights, (42) implies that per capita consumption should be equated across locations.

For the costates we have

$$v(x)U'\left(\frac{C(x,t)}{L}\right) = \lambda_K(x,t), \tag{45}$$

$$\mathbb{A}\Omega(\hat{T}(x,t))F'_q = \frac{\mu_R(x,t) - \beta\lambda_M(t)}{\lambda_K(x,t)}, \tag{46}$$

$$\frac{\dot{\lambda}_K(x,t)}{\lambda_K(x,t)} = \rho + \delta - \mathbb{A}\Omega(\hat{T}(x,t))F'_K, \tag{47}$$

$$\dot{\mu}_R(x,t) = \rho\mu_R(x,t), \tag{48}$$

$$\dot{\lambda}_M(t) = (\rho + m)\lambda_M(t) - \lambda_{T_0}(t)\frac{Z_1}{\left(1 + \frac{M(t)}{M_0}\right)}, \tag{49}$$

and

$$\dot{\lambda}_{T_0}(t) = (\rho + 1)\lambda_{T_0}(t) + \int_X v(x)L\Omega'_{C,T_0} - \int_X \lambda_K(x,t)\mathbb{A}\Omega'_{T_0}F dx. \tag{50}$$

The optimal temporal and latitudinal paths for the states, controls, and costates are determined by the solution of the welfare maximization problem, provided it exists and satisfies the desirable stability properties.

These paths can be written as

$$\begin{cases} \{K^*(t;D), K^*(t,x;D), R^*(t;D), M^*(t;D), T^*(t,x;D)\} & (51)\\ \{C^*(t;D), C^*(x,t;D), q^*(t;D), q^*(x,t;D)\} & (52)\\ \{\lambda^*_K(t;D), \lambda^*_M(t;D), \mu^*_R(t;D), \lambda^*_T(t,x;D)\} & (53) \end{cases}$$

where $(^*)$ denotes optimality.

Substituting (51) into (31) and (32) determines the optimal damages from climate change on a global or a location basis.

4.5 Competitive Equilibrium with Fossil Fuel Taxes

To determine the optimal mitigation policy for carbon emissions, Brock et al., (2012) proposed a worldwide decentralized market economy with private ownership containing respective consumers and firms at each latitude x and world fossil fuel firms.

Assume that each latitude x represents a country. In each country, the representative consumer maximizes utility subject to a permanent income constraint and takes $\Omega_C(T(x,t)) = \bar{\Omega}_C$ as parametric. The representative firm maximizes profits and takes world prices of fossil fuels, $p^C(x)$, and taxes on fossil fuel use, $\tau(x)$, as parametric. At the global scale, the world fossil fuel firms maximize profits and take taxes on their profits, θ, as parametric.

4.5.1 Consumers

The economy, at each time t, is populated by a finite number of infinitely lived agents which are referred to as "dynastic families." Moreover, it is assumed that all members of a given generation are alike. Time is continuous and a new dynasty is created at every time. As before, dynastic families take per capita damages, $\Omega_C(T(x,t)) = \bar{\Omega}_C$, as parametric beyond their control. They can also borrow and lend on the world bond markets at the rate of r, which for simplicity here is taken to be independent of t. The solution to the dynastic family problem makes use of the average human wealth and the stochastic discount factor that summarizes the influence of future prices on current decisions.

Consumers receive as lump-sum payments the fractions $s_F(x,t)$ and $s_{Tax}(x,t)$ of after-tax profits from fossil fuel firms, $\pi_F(t)$, and proceeds from fossil fuel taxes, $Tax(t)$, respectively. Moreover, the bonds, $B(x,t)$, and capital, $K(x,t)$, held at location x and time t must satisfy the constraints

$$B(x,0) = 0, \quad B(x,t)\exp(-rt) \underset{t\to\infty}{\rightarrow} 0 \text{ and } K(x,t)\exp(-rt) \underset{t\to\infty}{\rightarrow} 0. \tag{54}$$

The solvency constraint above only requires that the present discounted value of net non-monetary liabilities be zero in the long run.

The present value form of the consumer's budget constraint can then be written as

$$\int_0^\infty \exp(-rt)p^s(t)C(x,t)dt = K_0(x) + \int_0^\infty \exp(-rt)p^s(x)I(x,t)dt, \tag{55}$$

subject to capital and income constraints

$$K_0(x) = K(x,0) \tag{56}$$

and

$$I(x,t) = w(x,t)L + s_F(x,t)\pi_F(t) + s_{Tax}(x,t)Tax(t). \tag{57}$$

Here, $p^s(t)$ is the spot price of the consumption good at time t and $w(x,t)$ is the wage at location x.

The consumer thus solves the optimal control problem

$$\max_{\{C(x,t)\}} \int_0^\infty \exp(-\rho t)\left(LU\left(\frac{C(x,t)}{L}\right) - \bar{\Omega}_C\right)dt \tag{58}$$

with the optimality condition

$$U'\left(\frac{C(x,t)}{L}\right) = \Lambda(x)\exp(\rho t)p^C(t), \tag{59}$$

where $p^C(t) = \exp(-\rho t)p^s(t)$ and $\Lambda(x)$ is the Lagrangien multiplier for the permanent income constraint (55) and represents the marginal utility of capitalized income at location x.

We can then relate the equilibrium problem and the social planner's problem by letting $p^C(t) = \exp(-\rho t)\lambda_K(t;D)$. This produces the optimality condition

$$v(x)U'\left(\frac{C(x,t)}{L}\right) = \lambda_K(t;D). \tag{60}$$

Following the First Theorem of Welfare Economics, the welfare weights used by the social planner are the reciprocal of marginal utility, or Negishi weights (Stanton (2009)), $v(x) = 1/\Lambda(x)$.

The Second Theorem of Welfare Economics states that any solution of the social planner's problem for any arbitrary non-negative set of welfare weights across locations will satisfy the conditions for competitive equilibrium, except for the budget constraint in each location. These can only be satisfied through appropriate transfer payments across locations.

Thus, given a specific choice of welfare weights across locations, the corresponding solution to the social planner's problem can be implemented as a competitive equilibrium with transfers across locations.

4.5.2 Firms Producing Consumption Goods

The representative firm located at latitude x at time t solves the optimization problem

$$\begin{aligned}\max_{\{K(x,t),q(x,t)\}} & p^C(t)(\mathbb{A}\Omega(T(x,t))F(K(x,t),L,q(x,t)) \\ & -(r(t)+\delta)K(x,t)-L-(p(x,t)+\tau(x,t))q(x,t)),\end{aligned} \tag{61}$$

where $p(x,t)$ is the price paid for fossil fuels and τ is the carbon tax paid by the representative firm. Since $F(K,L,q)$ has constant returns to scale, the profits for firms producing consumption goods will be zero at each x.

The optimality conditions for capital, K, and carbon, q, yield

$$\mathbb{A}\Omega(T(x,t;D))F'_K(K(x,t),L,q(x,t)) = r(t)+\delta, \tag{62}$$

$$\mathbb{A}\Omega(T(x,t;D))F'_q(K(x,t),L,q(x,t)) = p(x,t)+\tau(x,t), \tag{63}$$

and

$$\mathbb{A}\Omega(T(x,t;D))F'_L(K(x,t),L,q(x,t)) = w(x,t). \tag{64}$$

Therefore, firms located at latitude x and time t will choose demands $K(x,t)$ and $q(x,t)$ according to (62) and (63).

To equate marginal value products across latitudes for all t, taxes on fossil fuels must be equal across locations. That is, $\tau(x,t)=\tau(t)$. If such a scenario arises through competition, then we also have $p(x,t)=p(t)$. This is addressed in Section 5.

4.5.3 Firms Producing Fossil Fuels

The world firms producing fossil fuels solve the optimization problem

$$\max_{\{q(x,t)\}} \int_0^\infty \exp(-rt)p^C(t)p(t)q(x,t)(1-\theta(t))dt, \tag{65}$$

subject to the world resource constraint

$$\int_0^\infty \int_X q(x,t)dxdt \le R_0, \tag{66}$$

and where, as before, $\theta(t)$ denotes the profit tax fossil fuel firms face.

The resulting first order condition is

$$p(t)(1-\theta(t)) = \mu_0 \exp(rt) = (\mathbb{A}\Omega F'_q - \tau(x,t))(1-\theta(t)), \tag{67}$$

where μ_0 is the Lagrange multiplier on the world resource constraint (66).

4.6 Equilibrium

From Section 4.5.2, we know that in any decentralized problem, the representative firms producing consumption goods located at latitude x will choose capital and carbon demands $K(x,t)$ and $q(x,t)$ so that

$$\mathbb{A}\Omega F'_K = r(t) + \delta, \tag{68}$$

$$\mathbb{A}\Omega F'_q = p(t) + \tau(x,t), \tag{69}$$

where F'_K and F'_q denote respectively the derivatives of F with respect to K and q. On the other hand, the market clearing conditions require

$$\begin{cases} \int_X K(x,t)dx = K(t), & \int_X q(x,t)dx = q(t), \quad \int_X B(x,t)dx = 0, \\ \int_X C(x,t)dx = C(t), & \text{and} \quad \int_X Y(x,t)dx = Y(t). \end{cases} \tag{70}$$

The temporal and latitudinal equilibrium paths for C, K, and q are determined by the consumer and market clearing conditions, as well as the optimality conditions (62)-(64) and (67) for a multiplier value $\bar{\mu}_0$ that exhausts the fossil fuels reserves (66).

Firms take temperature and taxes as parametric. Hence, these equilibrium paths can be written as

$$\{C^e(x,t;D,\tau,\theta,p), K^e(x,t;T,\tau,\theta,p), q^e(x,t;T,\tau,\theta,p)\}, \tag{71}$$

where $(^e)$ denotes equilibrium.

5 Optimal Mitigation Policy

Here we introduce a carbon tax, $\tau^*(x,t)$, to correct for the climate externality. Such a tax will induce consumers and firms to produce a competitive equilibrium equal to the Pareto optimal quantities. The social planner finds the optimal paths of (51) by solving the Pareto optimum problem, $\mathrm{PO}^*(v)$, for a given set of non-negative welfare weights. To achieve these optimal quantities, a tax is implemented through competitive markets. This ensures that both consumers and firms face a carbon tax equal to the social marginal cost $\tau^*(x,t)$ of carbon usage at each x and t. Implementation of the $\mathrm{PO}^*(v)$ by $\tau^*(x,t)$ is feasible by the concavity assumptions of the consumer and producer problems (Brock et al., (2012)).

Here we investigate the two cases of spatially uniform and spatially differentiated carbon taxes.

5.1 Spatially Uniform Optimal Carbon Taxes

Social and private marginal products for capital, K, and carbon, q, must be equated to implement $\mathrm{PO}^*(v)$. This is done by combining the welfare maximizing conditions (42)-(50) with the market equilibrium conditions (59), (62), (63), and (67). We denote the welfare maximizing paths by (*) and obtain

$$v(x)U'\left(\frac{C^*(x,t)}{L}\right) = \lambda_K^*(t;D), \tag{72}$$

$$\tau^*(x,t;D) = \frac{\mu_0^*(t;D)-\beta\lambda_M^*}{\lambda_K^*(t;D)} - p(t) = \frac{\mu_0^*(t;D)-\beta\lambda_M^*}{v(x)U'\left(\frac{C^*(x,t)}{L}\right)} - p(t), \tag{73}$$

and

$$p(t) = \frac{\mu_0 \exp(rt)}{1-\theta^*(t)}. \tag{74}$$

Suppose now that the social planner can implement without cost welfare weights of $\bar{v} = 1/\bar{\Lambda}$. Under such endowments, per capita consumption will be equated across latitudes, and the resulting Pareto optimum, PO($\bar{v}$), will also be a competitive equilibrium. The Second Theorem of Welfare Economics then states that PO($\bar{v}$) can be implemented with the appropriate transfers. In turn, (73) yields the optimal spatially uniform tax rate

$$\tau^*(t;D) = \frac{\mu_R^*(t;D)-\beta\lambda_M^*(t;D)}{\bar{v}U'(C_{\bar{v}}^*(t))/L} - p(t). \tag{75}$$

The climate externality is captured in the carbon tax by the costate variable $\lambda_M^*(t;D)$. In Appendix C.2 we show that $\lambda_M^*(t;D) < 0$. This implies that when we account for the climate externality, carbon taxes increase. In Section 6.2 we analyze the dependence of the tax functions on the thermal transport coefficient D.

5.2 Spatially Differentiated Optimal Carbon Taxes

In the absence of international transfers, a spatially uniform taxation policy is not possible, since it is not always feasible to equalize per capita consumption across locations. Thus, to correct for the climate externality, the social planner must implement a spatially differentiated taxation policy.

For the sake of analysis, we consider only the polar case where all locations are closed economies, with their own isolated capital markets, fossil fuel reserves, and fossil fuel markets. This is clearly unrealistic but it will allow us later to evaluate both qualitatively and quantitatively the forces that generate spatially differentiated carbon taxes.

From Section 4 we have the following competitive equilibrium conditions for the closed economies

$$U'\left(\frac{C(x,t)}{L}\right) = \Lambda(x)\exp(\rho t)p^C(x,t), \tag{76}$$

$$\mathbb{A}\Omega(\hat{T}(x,t))F_K' = r(x,t)+\delta, \tag{77}$$

$$\mathbb{A}\Omega(\hat{T}(x,t))F_q' = p(x,t)+\tau(x,t), \tag{78}$$

and

$$p(x,t) = \frac{\mu_0 \exp(-rt)}{1-\theta^*(x,t)} = \mu_0 \exp(-rt), \tag{79}$$

where we have set $\theta^*(x,t) = 0$ to simplify the exposition.

Denote by

$$p^*(x,t) = p(x,t) + \tau^*(x,t) \tag{80}$$

the social price of carbon at location x and time t. Then,

$$p^*(x,t,D) = \frac{\mu_R^*(x,t;D) - \beta\lambda_M^*(t;D)}{\lambda_K^*(x,t;D)} = \frac{\mu_R^*(x,t;D) - \beta\lambda_M^*(t;D)}{v(x)U'\left(\frac{C^*(x,t)}{L}\right)}, \tag{81}$$

where $(^*)$ indicates optimal paths.

If the social planner does not use Negishi weights, then $v(x)U'\left(\frac{C^*(x,t)}{L}\right) \neq v(x')U'\left(\frac{C^*(x',t)}{L}\right)$, and the optimal full social price of carbon is different across locations. The climate externality is again captured in the carbon tax by the costate variable $\lambda_M^*(t;D)$.

5.3 Welfare Weights

Assume that the social planner can utilize without cost the Negishi weights to implement a competitive equilibrium with zero transfers so that $v(x)U'\left(\frac{C^*(x,t)}{L}\right) = 1$. Then, spatially uniform optimal carbon taxes can be implemented with welfare weights

$$\bar{v} = \frac{1}{U'\left(\frac{C^*(x)}{L}\right)} = C^*(x). \tag{82}$$

5.3.1 Function for Welfare Weights

If, as discussed in Section 5.2, it is not possible to implement Negishi weights without cost, then the planner must implement spatially differentiated taxes to achieve the Pareto optimum allocation. Following Stanton (2009), we implement a policy that weights poorer countries by higher and wealthier countries by lower than the Negishi weights do in the spatially uniform case. As already mentioned, the integral of the weights over space x is normalized at unity.

We consider the simple stepwise weights:

$$v(x) = \begin{cases} \frac{1}{20} & \text{if} \quad -1 \leq x \leq -\frac{3}{4}, \\ \frac{17}{10}(x + \frac{3}{4}) + \frac{1}{20} & \text{if} \quad -\frac{3}{4} \leq x \leq -\frac{1}{4}, \\ \frac{9}{10} & \text{if} \quad -\frac{1}{4} \leq x \leq \frac{1}{4}, \\ \frac{17}{10}(\frac{3}{4} - x) + \frac{1}{20} & \text{if} \quad \frac{1}{4} \leq x \leq \frac{3}{4}, \\ \frac{1}{20} & \text{if} \quad \frac{3}{4} \leq x \leq 1. \end{cases} \tag{83}$$

Following the most recent world GDP mapping[6], we assume for this model that the GDP for the wealthiest countries is greater than the GDP for the poorest countries by a factor of ten. We also see in the model described above that GDP tends to grow away from the equator, as is the case with world GDP.

A more comprehensive function of welfare weights is investigated by Saez and Stantcheva (2013) but is beyond the scope of this paper.

[6]http://www.indexq.org/economy/gdp.php

6 Impact of Thermal Transportation and Endogenous Co-albedo

Next we make some additional simplifying assumptions. Namely, we assume no technological change, constant population, no fossil fuel constraint at each latitude, logarithmic utility function with no per capita damages in utility due to temperature increase, and a production function with constant returns to scale at each latitude x and time t.

The simplifications of no technological change and no population growth allow us to perform both a qualitative and quantitative analysis at the steady state. The no fossil fuel constraint implies $\mu_R = 0$ for all x and t, while the assumption about the utility function implies that $\Omega_C = 0$. We remark that intuitively, the assumption of no fossil fuel constraint is not unreasonable as a rising opportunity cost of fossil fuel extraction will guarantee that the stock of carbon is never actually exhausted.

6.1 Steady State

The steady state distributions for the socially optimal quantities of temperature, T^*, damages, $\Omega(-\gamma T^*)$, capital, $\bar{k}^*$, fossil fuels, $\bar{q}^*$, and consumption, $\bar{c}^*$, are derived in Appendix C and are listed below:

$$T^*_{[2],0} = Z_{[2],1} \ln\left(1 + \frac{M^*}{M_0}\right) + Z_{[2],0}, \tag{84}$$

$$T^*_{[4],0} = Z_{[4],1} \ln\left(1 + \frac{M^*}{M_0}\right) + Z_{[4],0}, \tag{85}$$

$$\hat{T}^*_{[2]}(x) = Z_{[2],0} + Z_{[2],1} \ln\left(1 + \frac{M^*}{M_0}\right) + \frac{Z_{[2],2}}{1+6D} P_2(x) + \frac{Z_{[2],4}}{1+20D} P_4(x), \tag{86}$$

and

$$\begin{aligned} \hat{T}^*_{[4]}(x) = & Z_{[4],0} + Z_{[4],1} \ln\left(1 + \frac{M^*}{M_0}\right) + \frac{Z_{[4],2}}{B+6D} P_2(x) \\ & + \frac{Z_{[4],4}}{B+20D} P_4(x) + \frac{Z_{[4],6}}{B+42D} P_6(x) + \frac{Z_{[4],8}}{B+72D} P_8(x), \end{aligned} \tag{87}$$

and

$$\bar{q}^*(x) = -\frac{(\rho+\delta)}{\beta\lambda^*_M} \frac{\alpha_q}{(\rho + (1-\alpha_K)\delta)} v(x) = (1+\rho)(\rho+m) \frac{\alpha_q}{\gamma\beta} \frac{(1+\frac{M^*}{M_0})}{Z_1} v(x), \tag{88}$$

$$\begin{aligned} \bar{k}^*(x) = & \left(\frac{\rho+\delta}{\alpha_K}\right)^{\frac{1}{\alpha_K - 1}} (\mathbb{A}\Omega(\hat{T}^*(x)))^{\frac{1}{1-\alpha_K}} \bar{q}^*(x)^{\frac{1}{1-\alpha_K}} \\ = & \Gamma_3 \Omega(\hat{T}^*(x))^{\frac{1}{1-\alpha_K}} = \Gamma_3 \exp\left(\frac{\gamma \hat{T}^*(x)}{1-\alpha_K}\right), \end{aligned} \tag{89}$$

$$\bar{c}^*(x) = \mathbb{A}\Omega(\hat{T}^*(x)) \bar{k}^*(x)^{\alpha_K} \bar{q}^*(x)^{\alpha_q} - \delta \bar{k}^*(x), \tag{90}$$

and

$$\lambda_K^*(x) = \frac{v(x)}{\bar{c}^*(x)}, \tag{91}$$

where

$$\Gamma_3 = \left(\frac{\rho+\delta}{\alpha_K}\right)\left(\mathbb{A}\bar{q}^*(x)\right)^{\frac{1}{1-\alpha_K}}. \tag{92}$$

6.2 Impact

Section 6.1 indicates that the steady state values of per capita capital and consumption at each location are affected by the heat transport coefficient D through their dependence on damages $\Omega(-\gamma\hat{T}^*(x))$. The impact of D on damages at latitude x can be determined by

$$\frac{\partial\Omega(x)}{\partial D} = \frac{\partial\Omega}{\partial\hat{T}^*}\frac{\partial\hat{T}^*(x)}{\partial D}. \tag{93}$$

Since $\frac{\partial\Omega}{\partial\hat{T}^*} < 0$, the impact of D on damages at a given latitude depends on the sign of derivative $\frac{\partial\hat{T}^*(x)}{\partial D}$.

For the two- and four-mode expressions of α and S, we find that the zeroes of $\frac{\partial\hat{T}^*(x)}{\partial D}$ are functions of D, which we call $x_{[2]}(D)$ and $x_{[4]}(D)$ for the two- and four-mode forms respectively. There are two zeroes in $[-1,1]$, which we call $x_{[2]}^{\pm}(D)$ and $x_{[4]}^{\pm}(D)$, for each form.

For the two-mode form, we see that

$$\frac{\partial x_{[2]}^{+}(D)}{\partial D} > 0 \quad \text{and} \quad \frac{\partial x_{[2]}^{-}(D)}{\partial D} < 0, \tag{94}$$

which implies that, as D increases, $|x_{[2]}(D)|$ increases.

For the four-mode form, we see that

$$\frac{\partial x_{[4]}^{+}(D)}{\partial D} < 0 \quad \text{and} \quad \frac{\partial x_{[4]}^{-}(D)}{\partial D} > 0, \tag{95}$$

which implies that, as D increases, $|x_{[4]}(D)|$ decreases.

We thus see a significant difference between the two- and four-modes in measuring the impact of thermal diffusion on damages.

The two-mode expression suggests that as the rate of diffusion increases, the area in the world for which temperature and damages are reduced, increases, while the area in the world for which temperature and damages are increased, decreases. The four-mode case suggests that as the rate of diffusion increases, the area in the world for which temperature and damages are reduced, decreases, while the area in the world for which temperature and damages are increased, increases.

Moreover we find that, for our particular choice of parameter values, we have $x_{[2]}(0) = \pm 0.5145$ and $\lim_{D\to\infty} x_{[2]}(D) = \pm 0.5707$ for the two-mode form, and $x_{[4]}(0) = \pm 0.6922$ and $\lim_{D\to\infty} x_{[4]}(D) = \pm 0.5877$ for the four-mode form.

We further notice that $\max_{\{D\}}|x_{[2]}(D)| < 1/\sqrt{3} < \min_{\{D\}}|x_{[4]}(D)|$. This implies that the approximation provided in Brock et al., (2012) overestimates the area in the world for which temperature and damages are reduced in the two-mode case, and underestimates the area in the world for which temperature and damages are reduced in the four-mode case.

Call x^* the positive zero of $\frac{\partial \hat{T}^*(x)}{\partial D}$ in $(0,1)$, where $\hat{T}^*(x)$ is the steady state distribution of temperature. Then, from (86) and (87), we find

$$\frac{\partial \hat{T}^*(x)}{\partial D} \begin{cases} = 0 & \text{for} \quad x = \pm x^*, \\ < 0 & \text{for} \quad -x^* < x < x^*, \\ > 0 & \text{for} \quad \begin{cases} x^* < x \leq 1, \\ -1 \leq x < -x^*, \end{cases} \end{cases} \tag{96}$$

and, thus,

$$\frac{\partial \Omega(x)}{\partial D} = \frac{\partial \Omega}{\partial \hat{T}^*} \frac{\partial \hat{T}^*(x)}{\partial D} \begin{cases} = 0 & \text{for} \quad x = \pm x^*, \\ > 0 & \text{for} \quad -x^* < x < x^* \qquad \text{damage reduction}, \\ < 0 & \text{for} \quad \begin{cases} x^* < x \leq 1, \\ -1 \leq x < -x^*, \end{cases} \qquad \text{damage increase}. \end{cases} \tag{97}$$

However, (89)-(91) yield

$$\frac{\partial \bar{k}^*}{\partial D} = \frac{1}{1-\alpha_K} \Gamma_3 \Omega^{\frac{\alpha_K}{1-\alpha_K}} \frac{\partial \Omega}{\partial \hat{T}^*} \frac{\partial \hat{T}^*(x)}{\partial D} \begin{cases} = 0 & \text{for} \quad x = \pm x^*, \\ > 0 & \text{for} \quad -x^* < x < x^*, \\ < 0 & \text{for} \quad \begin{cases} x^* < x \leq 1, \\ -1 \leq x < -x^*, \end{cases} \end{cases} \tag{98}$$

and

$$\frac{\partial \bar{c}^*}{\partial D} = \mathbb{A} \frac{\partial \Omega}{\partial \hat{T}^*} \frac{\partial \hat{T}^*(x)}{\partial D} \bar{k}^*(x)^{\alpha_K} \bar{q}^*(x)^{\alpha_q} + (\alpha_K \mathbb{A} \Omega \bar{k}^*(x)^{\alpha_K - 1} \bar{q}^*(x)^{\alpha_q} - \delta) \frac{\partial \bar{k}^*(x)}{\partial D}$$
$$\begin{cases} = 0 & \text{for} \quad x = \pm x^*, \\ > 0 & \text{for} \quad -x^* < x < x^*, \\ < 0 & \text{for} \quad \begin{cases} x^* < x \leq 1, \\ -1 \leq x < -x^*, \end{cases} \end{cases} \tag{99}$$

since $\alpha_K \mathbb{A} \Omega \bar{k}^*(x)^{\alpha_K - 1} \bar{q}^*(x)^{\alpha_q} - \delta > 0$ at the steady state due to (47), and

$$\frac{\partial \lambda_K^*(x)}{\partial D} = -\frac{v(x)}{\bar{c}^*(x)^2} \frac{\partial \bar{c}^*(x)}{\partial D} \begin{cases} = 0 & \text{for} \quad x = \pm x^*, \\ < 0 & \text{for} \quad -x^* < x < x^*, \\ > 0 & \text{for} \quad \begin{cases} x^* < x \leq 1, \\ -1 \leq x < -x^*. \end{cases} \end{cases} \tag{100}$$

Finally, it follows from (81) that the impact of D on the optimal social price of carbon at the steady state is

$$\frac{\partial \bar{p}^*}{\partial D} = \frac{\beta \lambda_M^*}{\lambda_K^*(x)^2} \frac{\partial \lambda_K^*(x)}{\partial D} \begin{cases} = 0 & \text{for} \quad x = \pm x^*, \\ > 0 & \text{for} \quad -x^* < x < x^*, \\ < 0 & \text{for} \quad \begin{cases} x^* < x \leq 1, \\ -1 \leq x < -x^*. \end{cases} \end{cases} \tag{101}$$

Let $I_1 = \{x : -x^* < x < x^*\}$ and $I_2 = \{x : -1 \leq x < -x^* \text{ and } x^* < x \leq 1\}$, i.e., I_1 are latitudes below $x = \pm x^*$, including the equator, while I_2 are latitudes above $x = \pm x^*$, including the North and South Pole.

Under the simplifying assumptions of Section 6, an increase in the heat transport coefficient D has the following effects on the steady state Pareto optimal solutions of the social planner's problem:

i) decreases temperature and damages, increases per capita capital and consumption, and increases the social cost of fossil fuels in I_1,

ii) increases temperature and damages, decreases per capita capital and consumption, and decreases the social cost of fossil fuels in I_2.

7 Qualitative Analysis

Here we see how changes in the output elasticities of capital and carbon affect the steady state solutions of the social planner problem in both the spatially uniform and spatially differentiated cases.

7.1 Impact of Carbon Output Elasticity

7.1.1 Stock of Atmospheric CO_2

Since, in view of (188), $M^* = \frac{M_0 C \alpha_q}{M_0 - C\alpha_q}$, where $C > 0$ and $C\alpha_q < M_0$,

$$\frac{\partial M^*}{\partial \alpha_q} = \frac{\partial}{\partial \alpha_q}\left(\frac{M_0 C \alpha_q}{M_0 - C\alpha_q}\right) = M_0 C \frac{(M_0 - C\alpha_q + C^2\alpha_q)}{(M_0 - C\alpha_q)} > 0. \tag{102}$$

This suggests that as the marginal cost of carbon decreases and a country shifts more towards carbon-heavy production, the steady state stock of man-made CO_2 in the atmosphere will increase. It follows from (86) and (87) (recall that $Z_{[n],1} > 0$ for $n \in \{2, 4\}$) that such a move will also increase the steady state value of temperature, $\hat{T}^*(x)$.

These results are intuitive as the greenhouse effect tells us that an increase in carbon emissions will increase temperature by blocking outgoing radiation.

7.1.2 Carbon Usage

Since, in view of (88),

$$\bar{q}^*(x) = (1+\rho)(\rho+m)\frac{\alpha_q}{\gamma\beta}\frac{(1+\frac{M^*}{M_0})}{Z_1}v(x), \tag{103}$$

it follows from (102) that $\frac{\partial \bar{q}^*}{\partial \alpha_q} > 0$.

This is obvious because as countries utilize more carbon in production, the steady state value for total fossil fuels used at each latitude will inevitably increase.

7.1.3 Spatially Uniform Optimal Carbon Taxes

From our simplifying assumptions, we find that the optimal tax on carbon in Section 5.1 is

$$\tau^* = -\beta\lambda_M^* - p, \tag{104}$$

where $-\beta\lambda_M^* > 0$. The impact of α_q on the optimal carbon tax is determined, in view of (102), by

$$\frac{\partial \tau^*(x)}{\partial \alpha_q} = \frac{\partial \tau^*}{\partial M^*}\frac{\partial M^*}{\partial \alpha_q}, \tag{105}$$

with $\frac{\partial M^*}{\partial \alpha_q} > 0$. Thus the impact of α_q on the optimal carbon tax depends on the sign of $\frac{\partial \tau^*}{\partial M^*}$.

It follows from (178) that

$$\begin{aligned}\frac{\partial \tau^*}{\partial M^*} &= \frac{\partial}{\partial M^*}(-\beta\lambda_M^*) \\ &= \frac{\partial}{\partial M^*}\left(\beta\frac{\gamma\Gamma_0}{(\rho+m)}\frac{Z_1}{(1+\frac{M^*}{M_0})}\right) \\ &= -\frac{\beta}{M_0}\frac{\gamma\Gamma_0}{(\rho+m)}\frac{Z_1}{(1+\frac{M^*}{M_0})^2} < 0,\end{aligned} \tag{106}$$

and

$$\frac{\partial p^*(x)}{\partial \alpha_q} = \frac{\partial}{\partial \alpha_q}(p+\tau^*(x)) < 0. \tag{107}$$

This suggests a decrease in both the optimal tax and optimal social price of carbon as countries move more towards carbon-heavy production in the spatially uniform problem. This makes sense because as α_q increases, the marginal cost of carbon in both a market and social setting decreases.

7.1.4 Spatially Differentiated Optimal Carbon Taxes

The optimal tax on carbon in Section 5.2 is

$$\begin{aligned}\tau^*(x) &= -\frac{\beta\lambda_M^*}{v(x)U'(\bar{c}^*(x))} - p(x) \\ &= -\frac{\beta\lambda_M^*\bar{c}^*(x)}{v(x)} - p(x),\end{aligned} \tag{108}$$

where $-\beta\lambda_M^* > 0$.

The impact of α_q on the optimal carbon tax is seen in

$$\begin{aligned}\frac{\partial}{\partial\alpha_q}\tau^*(x) &= \frac{\partial}{\partial\alpha_q}\left(-\frac{\beta\lambda_M^*}{v(x)\bar{c}^*(x)}\right)\\ &= \frac{\partial}{\partial\alpha_q}\left(\frac{\gamma\beta\Gamma_0}{(\rho+m)v(x)}\frac{Z_1}{(1+\frac{M^*}{M_0})}\frac{1}{\mathbb{A}\Omega(\hat{T}^*(x))\bar{k}^*(x)^{\alpha_K}\bar{q}^*(x)^{\alpha_q}-\delta\bar{k}^*(x)}\right) < 0,\end{aligned} \tag{109}$$

where the sign is determined from (102) and (103)

As with (107),

$$\frac{\partial}{\partial\alpha_q}p^*(x) = \frac{\partial}{\partial\alpha_q}(p+\tau^*(x)) < 0, \tag{110}$$

and, hence, the results match those of the spatially uniform case.

7.2 Impact of Capital Output Elasticity

7.2.1 Carbon Usage and Atmospheric Stock

The output elasticity α_K has no impact on the steady state distribution of carbon since neither M^* nor $\bar{q}^*(x)$ depend on it.

7.2.2 Capital Levels

Since, in view of (89),

$$\begin{aligned}\frac{\partial}{\partial\alpha_K}\bar{k}^*(x) =& \frac{\partial}{\partial\alpha_K}\left(\Gamma_3\exp\left(\frac{\gamma\hat{T}^*(x)}{1-\alpha_K}\right)\right)\\ =& \left(\frac{\partial}{\partial\alpha_K}\Gamma_3\right)\left(\frac{\gamma\hat{T}^*(x)}{1-\alpha_K}\right) + \Gamma_3\frac{\partial}{\partial\alpha_K}\exp\left(\frac{\gamma\hat{T}^*(x)}{1-\alpha_K}\right)\\ =& \left(\frac{\rho+\delta}{\alpha_K(\alpha_K-1)^2}\right)\Bigg(\left((\mathbb{A}\bar{q}^*(x))^{\frac{1}{1-\alpha_K}}\ln(\mathbb{A}\bar{q}^*(x))\right)\exp\left(\frac{\gamma\hat{T}^*(x)}{1-\alpha_K}\right)\\ &+ (\mathbb{A}\bar{q}^*(x))^{\frac{1}{1-\alpha_K}}\mathbb{A}\bar{q}^*(x)\exp\left(\frac{\mathbb{A}\bar{q}^*(x)}{1-\alpha_K}\right)\Bigg) > 0,\end{aligned} \tag{111}$$

it follows that, as the elasticity of capital increases, so does the steady state of capital across latitudes.

7.2.3 Spatially Uniform Optimal Carbon Taxes

Given (104), the impact of α_K on the optimal carbon tax is determined by the sign of

$$\begin{aligned}\frac{\partial \tau^*(x)}{\partial \alpha_K} &= \frac{\partial}{\partial \alpha_K}\left(\frac{\gamma\beta\Gamma_0}{(\rho+m)}\frac{Z_1}{(1+\frac{M}{M_0})}\right) \\ &= \frac{\partial}{\partial \alpha_K}\left(\frac{\gamma\beta}{(\rho+m)}\frac{1}{(1+\rho)}\frac{Z_1}{(1+\frac{M}{M_0})}\frac{(\rho+\delta)}{(\rho+(1-\alpha_K)\delta)}\right) \\ &= \frac{\partial}{\partial \alpha_K}\left(\frac{C}{\rho+(1-\alpha_K)\delta}\right) = \frac{\delta C}{\rho+(1-\alpha_K)\delta} > 0.\end{aligned} \tag{112}$$

with

$$C = \frac{\gamma\beta}{(\rho+m)}\frac{(\rho+\delta)}{(1+\rho)}\frac{Z_1}{(1+\frac{M}{M_0})} > 0. \tag{113}$$

In view of (80),

$$\frac{\partial}{\partial \alpha_q}p^*(x) = \frac{\partial}{\partial \alpha_q}(p+\tau^*(x)) > 0. \tag{114}$$

This suggests an increase in both the optimal tax and optimal social price of carbon as the output elasticity of capital increases in the spatially uniform problem, a fact which makes sense intuitively because, as α_K increases, the marginal cost of capital decreases. This results in an increase in the relative marginal cost of carbon in both a market and social setting.

7.2.4 Spatially Differentiated Optimal Carbon Taxes

In view of (108), the impact of α_K on the optimal carbon tax is determined by

$$\begin{aligned}\frac{\partial \tau^*(x)}{\partial \alpha_K} &= \frac{\partial}{\partial \alpha_K}\left(-\frac{\beta\lambda_M^*}{v(x)\bar{c}^*(x)}\right) \\ &= \frac{\partial}{\partial \alpha_K}\left(\frac{\tilde{C}}{(\rho+(1-\alpha_K)\delta)(\mathbb{A}\Omega(\hat{T}^*(x))\bar{k}^*(x)^{\alpha_K}\bar{q}^*(x)^{\alpha_q}-\delta\bar{k}^*(x))}\right) \\ &= \frac{(\mathbb{A}\Omega(\hat{T}^*(x))\bar{k}^*(x)^{\alpha_K-1}\bar{q}^*(x)-\delta)(\delta\bar{k}^*(x)-(\rho+(1-\alpha_K)\delta))}{((\rho+(1-\alpha_K)\delta)(\mathbb{A}\Omega(\hat{T}^*(x))\bar{k}^*(x)^{\alpha_K}\bar{q}^*(x)^{\alpha_q}-\delta\bar{k}^*(x)))^2}\tilde{C} > 0,\end{aligned} \tag{115}$$

with

$$\tilde{C} = \frac{\gamma\beta}{(\rho+m)}\frac{Z_1}{(1+\frac{M^*}{M_0})}\frac{(\rho+\delta)}{(1+\rho)}\frac{1}{v(x)}, \tag{116}$$

where the inequality above following from the fact that $\rho+(1-\alpha_K)\delta < \delta\bar{k}^*$.

As in (114),

$$\frac{\partial}{\partial \alpha_K}p^*(x) = \frac{\partial}{\partial \alpha_K}(p+\tau^*(x)) > 0, \tag{117}$$

and, hence, the results match those for the spatially uniform case.

8 Mode Comparison

We evaluate the robustness of the two- and four-mode expressions for the mean annual distribution of solar radiation energy and the co-albedo function. We utilize a distance metric between the two- and four-mode solutions to the optimal social planner's problems. More specifically, we compare the exact solutions to the temperature and damage problems, as well as the steady state quantities of capital and social price of carbon.

8.1 Method of Comparison

The L^2- or Euclidean norm provides an adequate measure of a distribution over a vector space. As we are interested in comparing the overall behavior of the two- and four-mode cases, we look for a way to relate the norms of each form. For mathematical simplicity, we use the square of the L^2-norm.

Denote by A_2 and A_4 the square of the L^2 norms for the two- and four-mode forms, respectively. The comparison statistic

$$R = \frac{|A_4 - A_2|}{A_2} \tag{118}$$

is used to evaluate the effectiveness of the two-mode form when compared to the four-mode form at the $\sigma = 0.05$ significance level. An R value close to 0 indicates that the two-mode form is an adequate estimate for the true distribution, while R values of 1 or greater indicate that the two-mode form either overestimates or underestimates the true distribution.

The test statistics in the following cases are all statistically significant, suggesting that the two-mode form for the mean annual distribution of solar radiation energy and the co-albedo function is adequate in modeling the steady state distributions of temperature, damages, capital in both the spatially uniform and spatially differentiated case, and the socially optimal price of carbon.

8.2 Reference Values

The following parameter values are determined empirically:

Table 2: Parameter Values

Parameter	**Value**	**Parameter**	**Value**
$\mathbb{A}$	50	β	1
ρ	0.05	δ	0.8
γ	0.0028	α_k	0.3
α_Q	0.08	α_L	1-0.3-0.08 = 0.62

8.3 Temperature

It follows from (86) that

$$A_2 = \int_X |\hat{T}^*_{[2]}(x)|^2 dx = \int_X \left| Z_{[2],0} + Z_{[2],1} \ln\left(1 + \frac{M^*}{M_0}\right) + \frac{Z_{[2],2}}{1+6D} P_2(x) + \frac{Z_{[2],4}}{1+20D} P_4(x) \right|^2 dx, \tag{119}$$

while (87) yields

$$\begin{aligned} A_4 &= \int_X |\hat{T}^*_{[4]}(x)|^2 dx \\ &= \int_X \left| Z_{[4],0} + Z_{[4],1} \ln\left(1 + \frac{M^*}{M_0}\right) + \frac{Z_{[4],2}}{B+6D} P_2(x) \right. \\ &\quad \left. + \frac{Z_{[4],4}}{B+20D} P_4(x) + \frac{Z_{[4],6}}{B+42D} P_6(x) + \frac{Z_{[4],8}}{B+72D} P_8(x) \right|^2 dx. \end{aligned} \tag{120}$$

Using the expressions for A_2 and A_4, for the above parameter values, we find using Mathematica that

$$R_T = \frac{|A_4 - A_2|}{A_2} = 0.0269. \tag{121}$$

8.4 Damages

It follows from (86) and (87) that

$$\begin{aligned} A_2 &= \int_X \left| \exp(-\gamma \hat{T}^*_{[2]}(x)) \right|^2 dx \\ &= \int_X \left| \exp\left(-\gamma \left(Z_{[2],0} + Z_{[2],1} \ln\left(1 + \frac{M^*}{M_0}\right) + \frac{Z_{[2],2}}{1+6D} P_2(x) + \frac{Z_{[2],4}}{1+20D} P_4(x) \right)\right) \right|^2 dx \end{aligned} \tag{122}$$

and

$$\begin{aligned} A_4 &= \int_X \left| \exp(-\gamma \hat{T}^*_{[4]}(x)) \right|^2 dx \\ &= \int_X \left| \exp\left(-\gamma \left(Z_{[4],0} + Z_{[4],1} \ln\left(1 + \frac{M^*}{M_0}\right) + \frac{Z_{[4],2}}{B+6D} P_2(x) \right.\right.\right. \\ &\quad \left.\left.\left. + \frac{Z_{[4],4}}{B+20D} P_4(x) + \frac{Z_{[4],6}}{B+42D} P_6(x) + \frac{Z_{[4],8}}{B+72D} P_8(x) \right)\right) \right|^2 dx. \end{aligned} \tag{123}$$

Again, from (122) and (123), for the above parameter values, the comparison statistic is

$$R_\Omega = \frac{|A_4 - A_2|}{A_2} = 0.0236. \tag{124}$$

8.5 Evolution of Capital

8.5.1 Spatially Uniform Problem

It follows from (89) that

$$\begin{aligned}
A &= \int_X |\bar{k}^*(x)|^2 dx \\
&= \int_X \left| \left(\frac{\rho+\delta}{\alpha_K}\right) (\bar{q}^*(x))^{\frac{1}{1-\alpha_K}} \exp\left(\frac{\gamma \hat{T}^*(x)}{1-\alpha_K}\right) \right|^2 dx \\
&= \int_X \left| \left(\frac{\rho+\delta}{\alpha_K}\right) \left((1+\rho)(\rho+m) \frac{\alpha_q}{2\gamma\beta} \frac{(1+\frac{M^*}{M_0})}{Z_1} \right)^{\frac{1}{1-\alpha_K}} \exp\left(\frac{\gamma \hat{T}^*(x)}{1-\alpha_K}\right) \right|^2 dx \\
&= \int_X \left| \Xi \exp\left(\frac{\gamma \hat{T}^*(x)}{1-\alpha_K}\right) \right|^2 dx,
\end{aligned} \tag{125}$$

with

$$\Xi = \left(\frac{\rho+\delta}{\alpha_K}\right) \left((1+\rho)(\rho+m) \frac{\alpha_q}{2\gamma\beta} \frac{(1+\frac{M^*}{M_0})}{Z_1} \right)^{\frac{1}{1-\alpha_K}}. \tag{126}$$

The third equality above follows from

$$\begin{aligned}
\bar{q}^*(x) &= -\frac{(\rho+\delta)}{\beta\lambda_M^*} \frac{\alpha_q}{(\rho+(1-\alpha_K)\delta)} \frac{v(x)}{\int_X v(x)dx} \\
&= (1+\rho)(\rho+m) \frac{\alpha_q}{\gamma\beta} \frac{(1+\frac{M^*}{M_0})}{Z_1} \frac{\bar{v}}{\int_X \bar{v}dx} \\
&= (1+\rho)(\rho+m) \frac{\alpha_q}{2\gamma\beta} \frac{(1+\frac{M^*}{M_0})}{Z_1},
\end{aligned} \tag{127}$$

which is derived from using the welfare weight $v(x) = \bar{v}$ in (88) for the spatially uniform problem.

Inserting (86) into (127), we get

$$\begin{aligned}
A_2 &= \int_X \left| \Xi \exp\left(\frac{\gamma \hat{T}^*_{[2]}(x)}{1-\alpha_K}\right) \right|^2 dx \\
&= \int_X \left| \Xi \exp\left(\frac{\gamma}{1-\alpha_K} \left(Z_{[2],0} + Z_{[2],1} \ln\left(1+\frac{M^*}{M_0}\right) + \frac{Z_{[2],2}}{1+6D} P_2(x) + \frac{Z_{[2],4}}{1+20D} P_4(x) \right)\right) \right|^2 dx,
\end{aligned} \tag{128}$$

while inserting (87) into (127) gives

$$
\begin{aligned}
A_4 &= \int_X \left| \Xi \exp\left(\frac{\gamma \hat{T}^*_{[4]}(x)}{1-\alpha_K} \right) \right|^2 dx \\
&= \int_X \left| \Xi \exp\left(\frac{\gamma}{1-\alpha_K} \left(Z_{[4],0} + Z_{[4],1} \ln\left(1 + \frac{M^*}{M_0}\right) + \frac{Z_{[4],2}}{B+6D} P_2(x) \right.\right.\right. \\
&\quad \left.\left.\left. + \frac{Z_{[4],4}}{B+20D} P_4(x) + \frac{Z_{[4],6}}{B+42D} P_6(x) + \frac{Z_{[4],8}}{B+72D} P_8(x) \right)\right)\right|^2 dx.
\end{aligned}
\tag{129}
$$

Using the parameter values with (128) and (129), we find

$$
R_{K,U} = \frac{|A_4 - A_2|}{A_2} = 0.00156. \tag{130}
$$

8.5.2 Spatially Differentiated Problem

We have

$$
\begin{aligned}
A &= \int_X |\bar{k}^*(x)|^2 dx \\
&= \int_X \left| \left(\frac{\rho+\delta}{\alpha_K} \right) (\bar{q}^*(x))^{\frac{1}{1-\alpha_K}} \exp\left(\frac{\gamma \hat{T}^*(x)}{1-\alpha_K} \right) \right|^2 dx \\
&= \int_X \left| \left(\frac{\rho+\delta}{\alpha_K} \right) \left((1+\rho)(\rho+m) \frac{\alpha_q}{\gamma\beta} \frac{(1+\frac{M^*}{M_0})}{Z_1} v(x) \right)^{\frac{1}{1-\alpha_K}} \exp\left(\frac{\gamma \hat{T}^*(x)}{1-\alpha_K} \right) \right|^2 dx \\
&= \int_X \left| \Xi \exp\left(\frac{\gamma \hat{T}^*(x)}{1-\alpha_K} \right) v(x)^{\frac{1}{1-\alpha_K}} \right|^2 dx,
\end{aligned}
\tag{131}
$$

with

$$
\Xi = \left(\frac{\rho+\delta}{\alpha_K} \right) \left((1+\rho)(\rho+m) \frac{\alpha_q}{\gamma\beta} \frac{(1+\frac{M^*}{M_0})}{Z_1} v(x) \right)^{\frac{1}{1-\alpha_K}}, \tag{132}
$$

where the third equality above follows from (88).

Using (83) and (86) into (131), we find, for the two-mode form,

$$\begin{aligned}
A_2 = &\int_{-1}^{-\frac{3}{4}} \left| \Xi \exp\left(\frac{\gamma}{1-\alpha_K} \hat{T}^*_{[2]}(x) \right) \left(\frac{1}{20} \right)^{\frac{1}{1-\alpha_K}} \right|^2 dx \\
&+ \int_{-\frac{3}{4}}^{-\frac{1}{4}} \left| \Xi \exp\left(\frac{\gamma}{1-\alpha_K} \hat{T}^*_{[2]}(x) \right) \left(\frac{17}{10} \left(x + \frac{3}{4} \right) + \frac{1}{20} \right)^{\frac{1}{1-\alpha_K}} \right|^2 dx \\
&+ \int_{-\frac{1}{4}}^{\frac{1}{4}} \left| \Xi \exp\left(\frac{\gamma}{1-\alpha_K} \hat{T}^*_{[2]}(x) \right) \left(\frac{9}{20} \right)^{\frac{1}{1-\alpha_K}} \right|^2 dx \\
&+ \int_{\frac{1}{4}}^{\frac{3}{4}} \left| \Xi \exp\left(\frac{\gamma}{1-\alpha_K} \hat{T}^*_{[2]}(x) \right) \left(\frac{17}{10} \left(\frac{3}{4} - x \right) + \frac{1}{20} \right)^{\frac{1}{1-\alpha_K}} \right|^2 dx \\
&+ \int_{\frac{3}{4}}^{1} \left| \Xi \exp\left(\frac{\gamma}{1-\alpha_K} \hat{T}^*_{[2]}(x) \right) \left(\frac{1}{20} \right)^{\frac{1}{1-\alpha_K}} \right|^2 dx.
\end{aligned} \tag{133}$$

Similarly, using (83) and (87) into (131), we find, for the four-mode form,

$$\begin{aligned}
A_4 = &\int_{-1}^{-\frac{3}{4}} \left| \Xi \exp\left(\frac{\gamma}{1-\alpha_K} \hat{T}^*_{[4]}(x) \right) \left(\frac{1}{20} \right)^{\frac{1}{1-\alpha_K}} \right|^2 dx \\
&+ \int_{-\frac{3}{4}}^{-\frac{1}{4}} \left| \Xi \exp\left(\frac{\gamma}{1-\alpha_K} \hat{T}^*_{[4]}(x) \right) \left(\frac{17}{10} \left(x + \frac{3}{4} \right) + \frac{1}{20} \right)^{\frac{1}{1-\alpha_K}} \right|^2 dx \\
&+ \int_{-\frac{1}{4}}^{\frac{1}{4}} \left| \Xi \exp\left(\frac{\gamma}{1-\alpha_K} \hat{T}^*_{[4]}(x) \right) \left(\frac{9}{20} \right)^{\frac{1}{1-\alpha_K}} \right|^2 dx \\
&+ \int_{\frac{1}{4}}^{\frac{3}{4}} \left| \Xi \exp\left(\frac{\gamma}{1-\alpha_K} \hat{T}^*_{[4]}(x) \right) \left(\frac{17}{10} \left(\frac{3}{4} - x \right) + \frac{1}{20} \right)^{\frac{1}{1-\alpha_K}} \right|^2 dx \\
&+ \int_{\frac{3}{4}}^{1} \left| \Xi \exp\left(\frac{\gamma}{1-\alpha_K} \hat{T}^*_{[4]}(x) \right) \left(\frac{1}{20} \right)^{\frac{1}{1-\alpha_K}} \right|^2 dx.
\end{aligned} \tag{134}$$

The comparison statistic is then

$$R_{K,D} = \frac{|A_4 - A_2|}{A_2} = 0.00156. \tag{135}$$

8.6 Spatially Differentiated Carbon Tax

We consider the socially optimal price for carbon under spatially differentiated tax policy with closed economies.

From (81), we have

$$\begin{aligned} A &= \int_X |p^*(x)|^2 dx \\ &= \int_X |p(x) + \tau^*(x)|^2 dx \\ &= \int_X \left| \frac{\beta \lambda_M^*}{v(x) U'\left(\bar{c}^*(x)\right)} \right|^2 dx \\ &= \int_X \left| \frac{\beta \lambda_M^*}{v(x) \bar{c}^*(x)} \right|^2 dx \\ &= \int_X \left| \frac{\bar{C}}{v(x)^{\frac{2-\alpha_K}{\alpha_K - 1}}} \exp\left(-\frac{\gamma \hat{T}^*(x)}{1-\alpha_K} \right) \right|^2 dx, \end{aligned} \tag{136}$$

with

$$\bar{C} = \left(\frac{\alpha_K}{\rho + (1-\alpha_K)\delta} \right)^2 (\mathbb{A}\alpha_q)^{\frac{1}{\alpha_K - 1}} (\gamma\beta)^{\frac{\alpha_K}{\alpha_K - 1}} \left((1+\rho)(\rho+m) \frac{(1 + \frac{M^*}{M_0})}{Z_1} \right)^{\frac{2-\alpha_K}{\alpha_K - 1}}. \tag{137}$$

Then,

$$\begin{aligned} A = &\int_{-1}^{-\frac{3}{4}} \left| \frac{\bar{C}}{(\frac{1}{20})^{\frac{2-\alpha_K}{\alpha_K-1}}} \exp\left(-\frac{\gamma \hat{T}^*(x)}{1-\alpha_K} \right) \right|^2 + \int_{-\frac{3}{4}}^{-\frac{1}{4}} \left| \frac{\bar{C}}{(\frac{17}{10}(x+\frac{3}{4}) + \frac{1}{20})^{\frac{2-\alpha_K}{\alpha_K-1}}} \exp\left(-\frac{\gamma \hat{T}^*(x)}{1-\alpha_K} \right) \right|^2 \\ &+ \int_{-\frac{3}{4}}^{-\frac{1}{4}} \left| \frac{\bar{C}}{(\frac{9}{10})^{\frac{2-\alpha_K}{\alpha_K-1}}} \exp\left(-\frac{\gamma \hat{T}^*(x)}{1-\alpha_K} \right) \right|^2 + \int_{\frac{1}{4}}^{\frac{3}{4}} \left| \frac{\bar{C}}{(\frac{17}{10}(\frac{3}{4} - x) + \frac{1}{20})^{\frac{2-\alpha_K}{\alpha_K-1}}} \exp\left(-\frac{\gamma \hat{T}^*(x)}{1-\alpha_K} \right) \right|^2 \\ &+ \int_{x\frac{3}{4}}^{1} \left| \frac{\bar{C}}{(\frac{1}{20})^{\frac{2-\alpha_K}{\alpha_K-1}}} \exp\left(-\frac{\gamma \hat{T}^*(x)}{1-\alpha_K} \right) \right|^2. \end{aligned} \tag{138}$$

It follows that

$$\begin{aligned} A_2 &= \int_X \left| \bar{C} v(x)^{\frac{2-\alpha_K}{\alpha_K-1}} \exp\left(-\frac{\gamma \hat{T}^*_{[2]}(x)}{1-\alpha_K} \right) \right|^2 dx \\ &= \int_X \left| \bar{C} v(x)^{\frac{2-\alpha_K}{\alpha_K-1}} \exp\left(-\frac{\gamma}{1-\alpha_K} \left(Z_{[2],0} + Z_{[2],1} \ln\left(1 + \frac{M^*}{M_0}\right) \right.\right.\right. \\ &\quad + \left.\left.\left. \frac{Z_{[2],2}}{1+6D} P_2(x) + \frac{Z_{[2],4}}{1+20D} P_4(x) \right) \right) \right|^2 dx, \end{aligned} \tag{139}$$

and

$$
\begin{aligned}
A_4 &= \int_X \left| \frac{\bar{C}}{v(x)^{\frac{2-\alpha_K}{\alpha_K - 1}}} \exp\left(-\frac{\gamma \hat{T}^*(x)}{1-\alpha_K} \right) \right|^2 dx \\
&= \int_X \left| \frac{\bar{C}}{v(x)^{\frac{2-\alpha_K}{\alpha_K - 1}}} \exp\left(-\frac{\gamma}{1-\alpha_K} \left(Z_{[4],0} + Z_{[4],1} \ln\left(1 + \frac{M^*}{M_0}\right) + \frac{Z_{[4],2}}{B+6D} P_2(x) \right.\right.\right. \\
&\quad \left.\left.\left. + \frac{Z_{[4],4}}{B+20D} P_4(x) + \frac{Z_{[4],6}}{B+42D} P_6(x) + \frac{Z_{[4],8}}{B+72D} P_8(x) \right)\right) \right|^2 dx.
\end{aligned} \tag{140}
$$

From (139) and (140), the comparison statistic is

$$
R_p = \frac{|A_4 - A_2|}{A_2} = 0.0474. \tag{141}
$$

9 Concluding Remarks

In this paper we have investigated the qualitative properties of the climate-economic problem introduced by Brock et al., (2012). We have performed a rigorous mathematical analysis for two- and four-mode expressions for the mean annual distribution of solar radiation energy and the co-albedo function, and have derived the exact solution to the temperature problem in both cases. We have used the exact solutions to analyze directly the impact of thermal diffusion and have concluded that thermal diffusion explicitly affects the spatial distributions of temperature and damages, but does so differently for the two- and four-mode cases.

Our results confirm that the transport of heat from lower latitudes to higher latitudes will increase temperature and damages there, while decreasing temperature and damages at the lower latitudes. If, as Pierce et al., (2011) suggest, human actions change the heat capacity of the oceans and atmosphere, then the diffusion coefficient is expected to change. Our results then become significant in showing the effect of human behavior on damages and other economic variables.

We have investigated the optimal mitigation policy introduced by Brock et al., (2012) to correct for the climate externality introduced in the coupled model. Our results confirm that when international transfers are allowed, a spatially uniform carbon tax can be implemented with Negishi transfer payments. When such payments are not allowed, we have found a spatially differentiated mitigation regime using a simple welfare weights function and have investigated the impact of heat diffusion for closed economies. While this polar case is by no means an adequate representation of the world economy, Brock et al., (2012) extend the results to show that under plausible assumptions, the social price of fossil fuels around the equator should be lower relative to other latitudes.

We have also provided results qualifying the effect of carbon and capital output elasticities on the steady state quantities. Our findings note specifically that, as the returns to scale of carbon increase, the socially optimal price of carbon decreases and the stock of man-made carbon in the atmosphere increases at the steady state. This is important in understanding the benefits (or lack thereof) of regulatory measures aimed at limiting the use of fossil fuels.

A possible next step would involve finding the specific carbon and capital output elasticities that minimize damages at the steady state.

In this paper we have also evaluated the robustness of the two- and four-mode expressions. Our results show that in the dynamic case, the two-mode form is adequate in modeling the steady state distributions when compared to the four-mode form. We believe that analyzing the strength of the two-mode form in the context of our model is one of the main contributions of our paper, since it allows for a better understanding of other climate-economic results.

Possible extensions to our work include a robustness analysis for economic models with variable labor and rates of depreciation, and the addition of an ocean layer or second-dimension (longitude) to the energy balance climate model. We see from Marshall and Rose (2009) that it is possible to add an ocean layer to the climate model, so a logical next step would be to perform the economic analyses of this paper with an adapted climate model. The results could then be compared to see the impact of the ocean or longitude on steady state distributions. The paper by Franning and Weaver (1996) investigates an energy and moisture balance model that has many of the same diffusion-type operators as the energy balance model in our paper. The methods that we present apply directly to their model as well, so the issue of robustness to the number of modes can be raised. These are both potentially interesting and important areas of further research in the field of climate-economics.

A

We derive the exact solution to the temperature problem for a two-mode form for the mean annual distribution of solar radiation energy and the co-albedo function.

We use the following properties of the Legendre polynomials

$$\frac{d}{dx}\left((1-x^2)\frac{d}{dx}P_n(x)\right) + n(n+1)P_n(x) = 0, \tag{142}$$

$$(n+1)P_{n+1}(x) = (2n+1)P_n(x) - nP_{n-1}(x), \tag{143}$$

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \langle P_n(x), P_m(x)\rangle = \frac{2\delta_{nm}}{2n+1}, \tag{144}$$

where

$$\delta_{nm} = 0 \text{ for } n \neq m,\ \delta_{nm} = 1 \text{ for } n = m, \tag{145}$$

to derive certain relations listed below.

A.1 Functional Form

The 2nd, 4th, 6th, and 8th order Legendre polynomials are, respectively,

$$P_2(x) = \frac{1}{2}(3x^2 - 1), \tag{146}$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \tag{147}$$

$$P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5), \tag{148}$$

and

$$P_8(x) = \frac{1}{128}(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35). \tag{149}$$

A.2 Derivations

Since

$$P_2^2(x) = \left(\frac{1}{2}(3x^2 - 1)\right)^2 = \frac{1}{4}(9x^4 - 6x^2 + 1), \tag{150}$$

it follows from (146) that

$$x^2 = \frac{1}{3}(2P_2 + 1). \tag{151}$$

Substituting (151) into the (147) we get

$$\begin{aligned} x^4 &= \frac{1}{35}(8P_4(x) + 30(\frac{1}{3}(2P_2(x) + 1)) - 3) \\ &= \frac{1}{35}(8P_4(x) + 20P_2(x) + 7). \end{aligned} \tag{152}$$

Substituting (151) and (152) into (150) gives

$$P_2^2(x) = \frac{18}{35}P_4(x) + \frac{2}{7}P_2(x) + \frac{1}{5}. \tag{153}$$

We use (7) and (153) to write $QS(x)\alpha(x)$ in (11) as

$$\begin{aligned} QS(x)\alpha(x) &= Q(\alpha_0 S_0 + (\alpha_0 S_2 + \alpha_2 S_0)P_2(x) + \alpha_2 S_2 P_2^2(x)) \\ &= Q\left(\alpha_0 S_0 + \alpha_2 S_2\left(\frac{18}{35}P_4(x) + \frac{2}{7}P_2(x) + \frac{1}{5}\right) + (\alpha_0 S_2 + \alpha_2 S_0)P_2(x)\right) \\ &= Q\left(\alpha_0 S_0 + \frac{18}{35}\alpha_2 S_2 P_4(x) + \left(\frac{2}{7}\alpha_0 S_2 + \alpha_2 S_0\right) P_2(x) + \frac{1}{5}\alpha_2 S_2\right). \end{aligned} \tag{154}$$

We write $\hat{T}(x,t) = \sum_{k\geq 0} T_{2k}P_{2k}(x)$. Using (142), (143), and (154) into (11) we find

$$\begin{aligned} &D\sum_{k\geq 0} 2k(2k+1)T_{2k}P_{2k}(x) + B\sum_{k\geq 0} T_{2k}P_{2k}(x) + A = \\ &h(t) + Q\left(\alpha_0 S_0 + \frac{18}{35}\alpha_2 S_2 P_4(x) + \left(\frac{2}{7}\alpha_0 S_2 + \alpha_2 S_0\right) P_2(x) + \frac{1}{5}\alpha_2 S_2\right). \end{aligned} \tag{155}$$

Multiplying (166) by $P_n(x)$ and integrating from -1 to 1, for each $n \in \{0, 2, 4\}$, we obtain

$$BT_0(t) = -A + h(t) + Q\left(\frac{1}{5}\alpha_2 S_2 + \alpha_0 S_0\right), \tag{156}$$

$$(B + 6D)T_2(D) = Q\left(\frac{2}{7}\alpha_2 S_2 + \alpha_0 S_2 + \alpha_2 S_0\right), \tag{157}$$

and

$$(B + 20D)T_4(D) = \frac{18}{35}\alpha_2 S_2, \tag{158}$$

while, for $k \geq 3$, $T_{2k} = 0$.

B

We derive the exact solution to the temperature problem for a four-mode form for the mean annual distribution of solar radiation energy and the co-albedo function.

It follows from (146) and (147) that

$$\begin{aligned} P_2(x)P_4(x) &= \frac{1}{2}(3x^2 - 1)\frac{1}{8}(35x^4 - 30x^2 + 3) \\ &= \frac{1}{16}(105x^6 - 125x^4 + 39x^2 - 3). \end{aligned} \tag{159}$$

Substituting (151) and (152) into the equation for $P_6(x)$ yields

$$\begin{aligned} x^6 &= \frac{1}{231}(16P_6(x) + 315x^4 - 105x^2 + 5) \\ &= \frac{1}{231}(16P_6(x) + \frac{315}{35}(8P_4(x) + 20P_2(x) + 7) - \frac{105}{3}(2P_2(x) + 1) + 5) \\ &= \frac{16}{231}P_6(x) + \frac{24}{77}P_4(x) + \frac{10}{21}P_2 + \frac{1}{7}. \end{aligned} \tag{160}$$

Substituting (151), (152), and (160) into (159) gives

$$\begin{aligned} P_2(x)P_4(x) =& \frac{1}{16}\left(105\left(\frac{16}{231}P_6(x)+\frac{24}{77}P_4(x)+\frac{10}{21}P_2+\frac{1}{7}\right)\right. \\ & \left. -\frac{125}{35}(8P_4(x)+20P_2(x)+7)+\frac{39}{3}(2P_2(x)+1)-3\right) \\ =& \frac{5}{11}P_6(x)+\frac{20}{77}P_4(x)+\frac{2}{7}P_2(x). \end{aligned} \tag{161}$$

Hence (147) gives

$$P_4^2(x) = \left(\frac{1}{8}(35x^4-30x^2+3)\right)^2 = \frac{1}{16}(1225x^8-2100x^6+110x^4-180x^2+9), \tag{162}$$

and, therefore,

$$x^8 = \frac{1}{6435}(128P_8(x)-12012x^6+6930x^4-1260x^2+35). \tag{163}$$

Using (151), (152), (160), and (163) in (162) yields

$$\begin{aligned} P_4^2(x) =& \frac{490}{1287}P_8(x)-\frac{1645}{24}\left(\frac{16}{231}P_6(x)+\frac{24}{77}P_4(x)+\frac{10}{21}P_2+\frac{1}{7}\right) \\ & +\frac{7895}{208}\left(\frac{1}{35}(8P_4(x)+20P_2(x)+7)\right)-\frac{7505}{1144}\left(\frac{1}{3}(2P_2(x)+1)\right)+\frac{10079}{41184} \\ =& \frac{490}{1287}P_8(x)-\frac{470}{99}P_6(x)-\frac{25401}{2002}P_4(x)-\frac{42475}{2772}P_2(x)-\frac{1193}{288}. \end{aligned} \tag{164}$$

We use (8), (153), (161), and (164) to write $QS(x)\alpha(x)$ in (11) as

$$\begin{aligned} QS(x)\alpha(x) =& Q(\alpha_0S_0+(\alpha_0S_2+\alpha_2S_0)P_2(x)+(\alpha_0S_4+\alpha_4S_0)P_4(x) \\ & +(\alpha_2S_4+\alpha_4S_2)P_2(x)P_4(x)+\alpha_2S_2P_2^2(x)+\alpha_4S_4P_4^2(x)) \\ =& Q\left(\alpha_0S_0+(\alpha_0S_2+\alpha_2S_0)P_2(x)+(\alpha_0S_4+\alpha_4S_0)P_4(x)\right. \\ & +(\alpha_2S_4+\alpha_4S_2)\left(\frac{5}{11}P_6(x)+\frac{20}{77}P_4(x)+\frac{2}{7}P_2(x)\right) \\ & +\alpha_2S_2\left(\frac{18}{35}P_4(x)+\frac{2}{7}P_2(x)+\frac{1}{5}\right) \\ & \left.+\alpha_4S_4\left(\frac{490}{1287}P_8(x)-\frac{470}{99}P_6(x)-\frac{25401}{2002}P_4(x)-\frac{42475}{2772}P_2(x)-\frac{1193}{288}\right)\right). \end{aligned} \tag{165}$$

We write $\hat{T}(x,t) = \sum_{k\geq 0} T_{2k}P_{2k}(x)$. Inserting (142), (143), and (165) in (11) we find

$$\begin{aligned} &D\sum_{k\geq 0} 2k(2k+1)T_{2k}P_{2k}(x) + B\sum_{k\geq 0} T_{2k}P_{2k}(x) + A = \\ &h(t) + Q\Bigg(\alpha_0 S_0 + (\alpha_0 S_2 + \alpha_2 S_0)P_2(x) + (\alpha_0 S_4 + \alpha_4 S_0)P_4(x) \\ &+ (\alpha_2 S_4 + \alpha_4 S_2)\left(\frac{5}{11}P_6(x) + \frac{20}{77}P_4(x) + \frac{2}{7}P_2(x)\right) \\ &+ \alpha_2 S_2\left(\frac{18}{35}P_4(x) + \frac{2}{7}P_2(x) + \frac{1}{5}\right) \\ &+ \alpha_4 S_4\left(\frac{490}{1287}P_8(x) - \frac{470}{99}P_6(x) - \frac{25401}{2002}P_4(x) - \frac{42475}{2772}P_2(x) - \frac{1193}{288}\right)\Bigg). \end{aligned} \tag{166}$$

Multiplying (166) by $P_n(x)$ and integrating from -1 to 1 for each $n \in \{0, 2, 4, 6, 8\}$ we get

$$BT_0(t) = -A + Q\left(\frac{1}{5}\alpha_2 S_2 - \frac{1193}{288}\alpha_4 S_4 + \alpha_0 S_0\right) + h(t), \tag{167}$$

$$(B+6D)T_2(D) = \frac{2}{7}(\alpha_2 S_4 + \alpha_4 S_2) - \frac{42475}{2772}\alpha_4 S_4 + \frac{2}{7}\alpha_2 S_2 + \alpha_0 S_2 + \alpha_2 S_0, \tag{168}$$

$$(B+20D)T_4(D) = \frac{20}{77}(\alpha_2 S_4 + \alpha_4 S_2) - \frac{25401}{2002}\alpha_4 S_4 + \frac{18}{35}\alpha_2 S_2 + \alpha_0 S_4 + \alpha_4 S_0, \tag{169}$$

$$(B+42D)T_6(D) = \frac{5}{11}(\alpha_2 S_4 + \alpha_4 S_2) - \frac{470}{99}\alpha_4 S_4, \tag{170}$$

$$(B+72D)T_8(D) = \frac{490}{1287}\alpha_4 S_4, \tag{171}$$

and $T_{2k} = 0$, for all $k \geq 5$.

C

We derive the steady state distributions for the socially optimal quantities.

C.1 Steady State

Let $\bar{k}(x) = \frac{K(x)}{L}, \bar{q}(x) = \frac{q(x)}{L}, \bar{c}(x) = \frac{C(x)}{L}$ denote per capita quantities. Then, the output per capita and the utility at each location are $\Omega(\hat{T}(x,t))\bar{k}(x)^{\alpha_k}\bar{q}(x)^{\alpha_q}$ and $\ln(\bar{c}(x))$, respectively.

To obtain the steady state, we set $\dot{\lambda}_K(x,t) = 0$ in (47) and divide by (46). Recalling that we set all the derivatives in (47)-(50) to be zero, we find

$$\frac{\alpha_K}{\alpha_q}\frac{\bar{q}(x)}{\bar{k}(x)} = -\frac{(\rho+\delta)}{\beta}\frac{\lambda_K}{\lambda_M} \tag{172}$$

and

$$\bar{q}(x) = -\frac{(\rho+\delta)}{\beta}\frac{\lambda_K}{\lambda_M}\frac{\alpha_q}{\alpha_K}\bar{k}(x). \tag{173}$$

It follows from (45) that

$$\lambda_K = \frac{v(x)}{\bar{c}(x)} \quad \text{and} \quad \bar{c}(x) = \mathbb{A}\Omega(\hat{T}(x))\bar{k}(x)^{\alpha_K}\bar{q}(x)^{\alpha_q} - \delta\bar{k}(x) \tag{174}$$

and, hence, in view of (172),

$$\bar{q}(x) = -\frac{(\rho+\delta)}{\beta\lambda_M}\frac{\alpha_q}{\alpha_K}\frac{v(x)}{\mathbb{A}\Omega(\hat{T}(x))\bar{k}(x)^{\alpha_K-1}\bar{q}(x)^{\alpha_q} - \delta\bar{k}(x)}. \tag{175}$$

On the other hand, (47) at the steady state yields

$$\mathbb{A}\Omega(\hat{T}(x))\bar{k}(x)^{\alpha_K-1}\bar{q}(x)^{\alpha_q} = \frac{(\rho+\delta)}{\alpha_K}. \tag{176}$$

From (3) the steady state stock of CO_2 is

$$M = \frac{\beta}{m}\int_X L\bar{q}(x)dx = \frac{\beta}{m}\int_X L\left(-\frac{(\rho+\delta)}{\beta\lambda_M}\frac{\alpha_q}{(\rho+(1-\alpha_K)\delta)}v(x)\right)dx. \tag{177}$$

C.2 Exponential Damage Function

Using $\Omega(\hat{T}(x,t)) = \exp\left(-\gamma\hat{T}(x,t)\right)$, so that $\Omega'_{T_0} = -\gamma\Omega(\hat{T}(x,t))$, and (48) - (50), we obtain at the steady state

$$\lambda_M^* = \frac{\lambda_{T_0}}{(\rho+m)}\frac{Z_1}{1+\frac{M^*}{M_0}} \tag{178}$$

and

$$\lambda_{T_0}^* = -\frac{\gamma}{(1+\rho)}\int_X \frac{v(x)}{\bar{c}(x)}(\mathbb{A}\Omega(\hat{T}(x))\bar{k}(x)^{\alpha_K}\bar{q}(x)^{\alpha_q})dx. \tag{179}$$

Since (47) and (50) yield

$$\bar{c}(x) = \bar{k}(x)(\mathbb{A}\Omega(\hat{T}(x))\bar{k}(x)^{\alpha_K-1}\bar{q}(x)^{\alpha_q} - \delta) \tag{180}$$

and

$$\alpha_K\mathbb{A}\Omega(\hat{T}(x))\bar{k}(x)^{\alpha_K-1}\bar{q}(x)^{\alpha_q} = \rho+\delta, \tag{181}$$

substituting into (179), we obtain the steady state value of $\lambda_{T_0}^*$

$$\lambda_{T_0}^* = -\frac{\gamma}{(1+\rho)}\int_X \frac{(\rho+\delta)}{(\rho+(1-\alpha_K)\delta)}v(x)dx = -\gamma\Gamma_0 < 0, \tag{182}$$

where

$$\Gamma_0 = \frac{1}{(1+\rho)}\frac{(\rho+\delta)}{(\rho+(1-\alpha_K)\delta)}\int_X v(x)dx > 0. \tag{183}$$

Then,

$$\lambda_M^* = -\frac{\gamma\Gamma_0}{(\rho+m)}\frac{Z_1}{(1+\frac{M^*}{M_0})} < 0. \tag{184}$$

Since λ_M^* does not depend on x, using (184) into (177), we can find the steady state value

$$M^* = -\frac{1}{m\lambda_M^*}\Gamma_1, \tag{185}$$

where

$$\Gamma_1 = \int_X \frac{\alpha_q(\rho+\delta)L}{(\rho+(1-\alpha_K)\delta)} v(x)dx, \tag{186}$$

which are both independent of D.

Combining (184) and (185) we get

$$\begin{aligned} M^* &= \frac{1}{m}\frac{(\rho+m)}{\gamma}\frac{(1+\frac{M^*}{M_0})}{Z_1}\frac{\Gamma_1}{\Gamma_0} \\ &= \left(1+\frac{M^*}{M_0}\right)\frac{(\rho+m)}{m\gamma}\frac{1}{Z_1}(1+\rho)\alpha_q L, \end{aligned} \tag{187}$$

which implies

$$M^* = \frac{M_0\Gamma_2}{M_0-\Gamma_2}, \tag{188}$$

with

$$\Gamma_2 = \frac{(\rho+m)}{m\gamma}\frac{(1+\rho)}{Z_1}\alpha_q L, \tag{189}$$

subject to the constraint

$$1-\frac{\Gamma_2}{M_0} > 0. \tag{190}$$

This condition is satisfied by the parameter values in Table 2.

Combining all of the above results, we deduce that the steady state values for the rest of the variables are

$$T^*_{[2],0} = Z_{[2],1}\ln\left(1+\frac{M^*}{M_0}\right) + Z_{[2],0}, \tag{191}$$

$$T^*_{[4],0} = Z_{[4],1}\ln\left(1+\frac{M^*}{M_0}\right) + Z_{[4],0}, \tag{192}$$

$$\hat{T}^*_{[2]}(x) = Z_{[2],0} + Z_{[2],1}\ln\left(1+\frac{M^*}{M_0}\right) + \frac{Z_{[2],2}}{1+6D}P_2(x) + \frac{Z_{[2],4}}{1+20D}P_4(x), \tag{193}$$

$$\begin{aligned} \hat{T}^*_{[4]}(x) &= Z_{[4],0} + Z_{[4],1}\ \ln\left(1+\frac{M^*}{M_0}\right) + \frac{Z_{[4],2}}{B+6D}P_2(x) \\ &\quad + \frac{Z_{[4],4}}{B+20D}P_4(x) + \frac{Z_{[4],6}}{B+42D}P_6(x) + \frac{Z_{[4],8}}{B+72D}P_8(x) \end{aligned} \tag{194}$$

and

$$\bar{q}^*(x) = -\frac{(\rho+\delta)}{\beta\lambda_M^*}\frac{\alpha_q}{(\rho+(1-\alpha_K)\delta)}v(x) = (1+\rho)(\rho+m)\frac{\alpha_q}{\gamma\beta}\frac{(1+\frac{M^*}{M_0})}{Z_1}v(x), \tag{195}$$

$$\begin{aligned}\bar{k}^*(x) =& \left(\frac{\rho+\delta}{\alpha_K}\right)^{\frac{1}{\alpha_K-1}} (\mathbb{A}\Omega(\hat{T}^*(x)))^{\frac{1}{1-\alpha_K}}\bar{q}^*(x)^{\frac{1}{1-\alpha_K}} \\ =& \Gamma_3\Omega(\hat{T}^*(x))^{\frac{1}{1-\alpha_K}} = \Gamma_3\exp\left(\frac{\gamma\hat{T}^*(x)}{1-\alpha_K}\right),\end{aligned} \tag{196}$$

$$\bar{c}^*(x) = \mathbb{A}\Omega(\hat{T}^*(x))\bar{k}^*(x)^{\alpha_K}\bar{q}^*(x)^{\alpha_q} - \delta\bar{k}^*(x), \tag{197}$$

$$\lambda_K^*(x) = \frac{v(x)}{\bar{c}^*(x)}, \tag{198}$$

where

$$\Gamma_3 = \left(\frac{\rho+\delta}{\alpha_K}\right)(\mathbb{A}\bar{q}^*(x))^{\frac{1}{1-\alpha_K}}. \tag{199}$$

Using the specific values of the coefficients leads to the signs

$$Z_{[2],0}, Z_{[2],1}, Z_{[2],4} > 0, Z_{[2],2} < 0 \tag{200}$$

and

$$Z_{[4],0}, Z_{[4],1}, Z_{[4],8} > 0, Z_{[4],2}, Z_{[4],4}, Z_{[4],6} < 0. \tag{201}$$

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About RDCEP

The Center brings together experts in economics, physical sciences, energy technologies, law, computational mathematics, statistics, and computer science to undertake a series of tightly connected research programs aimed at improving the computational models needed to evaluate climate and energy policies, and to make robust decisions based on outcomes.

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For more information please contact us at
info-RDCEP@ci.uchicago.edu
or visit our website:
www.rdcep.org

RDCEP
Computation Institute
University of Chicago
5735 S. Ellis Ave.
Chicago, IL, 60637 USA
+1 (773) 834 1726